Introduction to Multipartite Quantum Entanglement and Quantum Nonlocality

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Part I

Introduction to Multipartite Quantum Entanglement

quantum entanglement

physical aspect

quantum entanglement
= a type of very strong correlations arising from the quantum formalism that cannot be explained within a classical framework

$\longrightarrow$ spatially separated systems appear to “communicate” instantaneously

a full understanding of quantum entanglement requires knowledge of the theory of quantum measurement

$\longrightarrow$ consistency with the special theory of relativity is preserved!!!

entanglement = one of the resources necessary for quantum computation

...

plan

- linear (quantum) algebra
- tensor product and partial operations
- pure and mixed quantum state formalism
- purification procedure
- Schmidt decomposition (SVD)

complex (linear) Hilbert space with an inner product

mathematical notation

$$\mathcal{H} \quad \cong \quad \mathbb{C}^d \quad \cong \quad \mathbb{R}^d \times \mathbb{R}^d \quad : \quad d < \infty$$

vector $=$ <u>normalized</u> element of the vector space $\mathcal{H}$
a vector of dimension d has d complex components $z_j = x_j + iy_j \ : \ i = \sqrt{-1}$

$$\mathbf{z} \in \mathcal{H} \qquad : \qquad \mathbf{z} = \begin{bmatrix} z_1 \\ \vdots \\ z_d \end{bmatrix} = [z_1, ..., z_d]^{\mathrm{T}} =: |z\rangle \qquad \mathbf{z}^{\mathrm{T}*} =: \langle z|$$

$$\mathbf{a}+\mathbf{b} = \begin{bmatrix} a_1 \\ \vdots \\ a_d \end{bmatrix} + \begin{bmatrix} b_1 \\ \vdots \\ b_d \end{bmatrix} =: |a+b\rangle \qquad \alpha\mathbf{a} = \alpha \begin{bmatrix} a_1 \\ \vdots \\ a_d \end{bmatrix} = \begin{bmatrix} \alpha a_1 \\ \vdots \\ \alpha a_d \end{bmatrix} =: \alpha|a\rangle : \alpha \in \mathbb{C}$$

inner (scalar) product

multiplication of two vectors

dimensional compatibility required!

$$\mathbf{a}\cdot\mathbf{b} \equiv \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_d \end{bmatrix}^* \cdot \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_d \end{bmatrix} = a_1^* b_1 + a_2^* b_2 + \ldots + a_d^* b_d = \langle a|b\rangle = \sum_{j=1}^{d} a_j^* b_j \in \mathbb{C}$$

antilinearity in the first argument (physics convention)

perpendicularity of vectors = orthogonality[1]

$$\mathbf{a} \perp \mathbf{b} \quad \Longleftrightarrow \quad \mathbf{a}\cdot\mathbf{b} = 0 \quad \Longleftrightarrow \quad \langle a|b\rangle = 0$$

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[1] normalized vectors $\Longrightarrow$ ortho<u>normal</u>ity

vector norm

every nonzero vector $\mathbf{z} \in \mathcal{H}$ has a direction, orientation, and magnitude[2]

magnitude = vector length = vector norm (Frobenius norm)

$$||\mathbf{z}||_{\mathrm{F}} \equiv \sqrt{\sum_{j=1}^{d} |z_j|^2} = \sqrt{\langle z|z\rangle}$$

we consider only normalized vectors, i.e., $||\mathbf{z}||_{\mathrm{F}} = 1$

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[2] direction and orientation are discussed together with the Bloch sphere

orthonormal basis of a vector space (ONB)

every vector $\mathbf{z} \in \mathcal{H}$ (its coordinates) can be expressed uniquely using a special set of vectors called a basis of the Hilbert space $\mathcal{H}$

by choosing a basis we choose a representation of all vectors in $\mathcal{H}$

when changing between different bases we only change the way the vector coordinates are written – some bases simplify (significantly) the form of vectors

the special (canonical) computational basis has the form

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} =: |0\rangle, \qquad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} =: |1\rangle, \qquad \dots, \qquad \mathbf{e}_d = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} =: |d-1\rangle$$

thus $\mathbf{z} = z_1\mathbf{e}_1 + z_2\mathbf{e}_2 + \dots + z_d\mathbf{e}_d =: \sum_j z_j|j\rangle \, : \, z_j \in \mathbb{C}$ are the coordinates of the vector $\mathbf{z} = |z\rangle$

a basis is a reference frame used in quantum measurement

outer product

another type[3] of multiplication of two vectors

$$\mathbf{ab}^\dagger = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_d \end{bmatrix} \begin{bmatrix} b_1^* \\ b_2^* \\ \vdots \\ b_d^* \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_d \end{bmatrix} [b_1^*, b_2^*, \ldots, b_d^*] = |a\rangle\langle b|$$

$$= \begin{bmatrix} a_1 b_1^* & a_1 b_2^* & \ldots & a_1 b_d^* \\ a_2 b_1^* & a_2 b_2^* & \ldots & a_2 b_d^* \\ \vdots & \vdots & & \vdots \\ a_d b_1^* & a_d b_2^* & \ldots & a_d b_d^* \end{bmatrix} \in \mathbb{C}^{d \times d}$$

as a result we obtain a rank-one matrix

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[3] we will soon encounter a third way of multiplying vectors

Dirac notation

physical notation

vector $\mathbf{z} \to |z\rangle \in \mathcal{H}$

conjugate vector $\mathbf{z}^\dagger \to \langle z| \in \mathcal{H}^*$

basis $\mathbf{e}_j \to |j\rangle$

inner (scalar) product $\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^\dagger \mathbf{b} \to \langle a|b\rangle \in \mathbb{C}$

outer product $\mathbf{a}\mathbf{b}^\dagger \to |a\rangle\langle b| \in \mathbb{C}^{d \times d}$

tensor product of vectors $\mathbf{a} \otimes \mathbf{b} \to |a\rangle \otimes |b\rangle \equiv |a\rangle|b\rangle = |ab\rangle \in \mathbb{C}^{d_1 d_2}$

spectral decomposition of a Hermitian matrix

$$A = \sum_{j=1}^{d} a_j \lambda_j a_j^\dagger \to A = \sum_a a|a\rangle\langle a|$$

vector → quantum state (wavefunction) $\mathbf{z} \to |\psi\rangle$

physical system state

physical notation

pure state = element of the Hilbert space $\mathcal{H}$

complete information about the physical system

$$|\psi\rangle = |0\rangle$$

$$|\psi\rangle = \frac{1}{\sqrt{3}}|0\rangle + \frac{1}{\sqrt{3}}|1\rangle + \frac{1}{\sqrt{3}}|2\rangle$$

$$|\psi\rangle = \frac{i}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

mixed state = element of the operator space acting on $\mathcal{H}$

statistical mixture of pure states representing incomplete information about the system

$$\rho = |\psi\rangle\langle\psi| = \text{projector onto the subspace generated by } |\psi\rangle$$

$$\rho = \sum_{j=1}^{r} p_j |j\rangle\langle j| \quad : \quad p_j \in [0,1] \quad \wedge \quad \sum_{j=1}^{r} p_j = 1 \quad : \quad r \in [1,d]$$

density matrix $\rho \in \mathbb{C}^{d\times d}$

normalization[4]

$$\mathrm{Tr}(\rho) = \sum_{(j=k)=1}^{d} \rho_{j,k} = 1$$

Hermiticity[5]

$$\rho = \rho^\dagger$$

positive semi-definiteness[6] (PSD)

$$\rho \geqslant 0 \quad \Longleftrightarrow \quad \mathrm{eig}(\rho) \geqslant 0$$

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[4] trace = sum of the diagonal elements of the matrix

[5] Hermitian operator = valid quantum observable measurable in the laboratory

[6] eigenvalue = allowed outcome of a measurement of a physical quantity → q. measurement

pure/(mixed) states

- $\psi = |\psi\rangle\langle\psi|$ = projector (pure state), $|\psi\rangle \in \mathbb{C}^d$
- classical statistical mixture of pure states

$$\rho = \sum_{j=0}^{r-1} p_j |\psi_j\rangle\langle\psi_j| \qquad : \qquad p_j = \text{some probability distribution}$$

- **maximally mixed** state, when $\forall_{j\in[d]} : p_j = 1/d$ and $\forall_j : |\psi_j\rangle = |j\rangle$ (ONB)

  second condition means there is no preferred direction in the space

$$\rho_* = \sum_{j=0}^{d-1} \frac{1}{d} |j\rangle\langle j| = \frac{1}{d}\mathbb{I}_d$$

purity = measure of the mixedness of a quantum state

$$\mathcal{P}(\rho) \equiv \mathrm{Tr}\{\rho^2\} \in \left[\frac{1}{d}, 1\right]$$

matrix (operator)

matrix (array of numbers) = numerical representation of a linear operation which, when acting on a vector, transforms it into another vector[7]

$$A\,\mathbf{z} = \mathbf{z}'$$

we consider only square matrices

$$A = [a_{j,k}] = \begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,d} \\ a_{2,1} & a_{2,2} & \dots & a_{2,d} \\ \vdots & \vdots & & \vdots \\ a_{d,1} & a_{d,2} & \dots & a_{d,d} \end{bmatrix} \in \mathbb{C}^{d\times d} \qquad : \qquad a_{j,k} \in \mathbb{C}$$

a special case is the identity matrix formed from the computational basis vectors

$$\mathbb{I}_d = [\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_d] = \begin{bmatrix} 1 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 1 \end{bmatrix} = \sum_{j=0}^{d-1} |j\rangle\langle j| \quad : \quad \{|j\rangle\}_{j=0}^{d-1} = \text{ONB}$$

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[7] similarly to vectors, the form of the matrix depends on the choice of basis!

unitary matrix

unitary matrix[8]

$$U \in \mathbb{U}(d) \quad : \quad UU^\dagger = U^\dagger U = \mathbb{I}_d$$

is a special case of a isometry, and thus preserves the norm of a vector

$$||\mathbf{z}||_{\mathrm{F}} = 1 \quad \Longrightarrow \quad ||U\mathbf{z}||_{\mathrm{F}} = 1$$

$$U = U(\varphi, \alpha, \beta, \gamma) = e^{i\varphi} \begin{bmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{bmatrix} \begin{bmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{bmatrix} \begin{bmatrix} e^{i\gamma} & 0 \\ 0 & e^{-i\gamma} \end{bmatrix}$$

a unitary operation transforms one ONB of a Hilbert space into another ONB

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[8] the Hermitian conjugate of a matrix should be understood as the conjugation of all its columns treated as vectors, i.e., rows become complex-conjugated columns

“trace trick”

let $M \in \mathbb{C}^{d\times d}$

$$\mathrm{Tr}(M) = \sum_{j=0}^{d-1} M_{j,j} = \sum_{j=0}^{d-1} \langle j|M|j\rangle \quad : \quad \{|j\rangle\} = \mathrm{ONB}$$

let $M = |j\rangle\langle k|$

$$\mathrm{Tr}\Big\{|j\rangle\langle k|\Big\} = \sum_{l=0}^{d-1} \langle l|j\rangle\langle k|l\rangle = \sum_{l=0}^{d-1} \langle k|l\rangle\langle l|j\rangle = \langle k| \left(\sum_{l=0}^{d-1} |l\rangle\langle l|\right) |j\rangle = \langle k|j\rangle$$

tensor product (Kronecker product)

= an operation that mathematically describes a quantum system composed of multiple components (at least two)

$$A \otimes B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \begin{bmatrix} e & f & g \\ h & i & j \\ k & l & m \end{bmatrix} \equiv \left[\begin{array}{c|c} aB & bB \\ \hline cB & dB \end{array}\right] =$$

$$= \left[\begin{array}{ccc|ccc} ae & af & ag & be & bf & bg \\ ah & ai & aj & bh & bi & bj \\ ak & al & am & bk & bl & bm \\ \hline ce & cf & cg & de & df & dg \\ ch & ci & cj & dh & di & dj \\ ck & cl & cm & dk & dl & dm \end{array}\right]$$

in this way we use ⊗ for arbitrary matrices and/or vectors, creating objects with a distinguished block structure

here: 4 blocks of size 3 × 3...

$\otimes$ properties

- $(A \otimes B)(C \otimes D) = AC \otimes BD$
- $(A \otimes B)(|a\rangle \otimes |b\rangle) = A|a\rangle \otimes B|b\rangle = A \otimes B|ab\rangle$
- $(|a\rangle + |b\rangle) \otimes |c\rangle = |ac\rangle + |bc\rangle$
- $|a\rangle \otimes (|b\rangle + c|\rangle) = |ab\rangle + |ac\rangle$
- $(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$
- $(|a\rangle \otimes |b\rangle)^\dagger = \langle a| \otimes \langle b|$
- $(|a\rangle \otimes |b\rangle) \cdot (|c\rangle \otimes |d\rangle) = |ac\rangle \otimes |bd\rangle$
- $(\langle k| \otimes \langle l|) \cdot (|i\rangle \otimes |j\rangle) = \langle k|i\rangle \otimes \langle l|j\rangle = \langle k|i\rangle\langle l|j\rangle$
- $|ab\rangle\langle cd| = (|a\rangle \otimes |b\rangle)(\langle c| \otimes \langle d|) = |a\rangle\langle c| \otimes |b\rangle\langle d|$
- $|\psi\rangle \otimes |\varphi\rangle = |\psi\rangle|\varphi\rangle = |\psi\varphi\rangle = |\psi, \varphi\rangle = \ldots$
- $\mathrm{Tr}(A \otimes B) = \mathrm{Tr}(A) \cdot \mathrm{Tr}(B)$
- ...

- $A \otimes B = C \not\Longrightarrow B = A^{-1} \otimes C$

multipartite quantum systems (tensor product $\otimes$)

Hilbert space of a multipartite quantum system
= tensor product of the spaces corresponding to the subsystems

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C \otimes \ldots = \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B} \otimes \mathbb{C}^{d_C} \otimes \ldots$$

a finite-dimensional space is only a mathematical concept

the true physical space has “infinitely” many dimensions, of which we can sometimes <u>isolate</u> a certain fragment …

$$\mathbb{C}^d \approx \mathbb{C}^d_{\mathtt{laboratory}} \otimes \mathcal{H}_{\mathtt{environment}} \Longrightarrow \text{“qudits” do not exist in Nature!}$$

… and then we take the average over the remaining degrees of freedom

we will return to multipartite systems when discussing quantum entanglement

state vector – multi-system formalism

let $\{|j\rangle\}$ be an ONB in $\mathbb{C}^d$

$$|\psi\rangle \in \mathbb{C}^d \quad \Longrightarrow \quad |\psi\rangle = \sum_{j=0}^{d-1} \psi_j |j\rangle = \sum_{j=0}^{d-1} \underbrace{\langle j|\psi\rangle}_{\psi_j \in \mathbb{C}} |j\rangle$$

if $\{|j_k\rangle\}$ is an ONB in the space $\mathbb{C}^{d_k}$, then a natural ONB of the space

$$\mathcal{H} = \bigotimes_{k=1}^{N} \mathbb{C}^{d_k} \text{ is the tensor product } \Big\{|j_1\rangle \otimes |j_2\rangle \otimes \ldots \otimes |j_N\rangle\Big\}$$

życzenia na 2026r.

generally

$$|\psi\rangle \in \bigotimes_{k=1}^{N} \mathbb{C}^{d_k} \quad \Longrightarrow \quad |\psi\rangle = \sum_{j_1=0}^{d_1-1} \ldots \sum_{j_N=0}^{d_N-1} \underbrace{\psi_{j_1,\ldots,j_N}}_{\text{tensor}} |j_1\rangle \otimes \ldots \otimes |j_N\rangle$$

partial operations – partial trace

let $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

$$\mathrm{Tr}_A(A \otimes B) \equiv \mathrm{Tr}(A) \otimes B = \mathrm{Tr}(A) \cdot B$$

$$\mathrm{Tr}_B(A \otimes B) \equiv A \otimes \mathrm{Tr}(B) = A \cdot \mathrm{Tr}(B)$$

$$\mathrm{Tr}(A \otimes B) = \mathrm{Tr}_A \mathrm{Tr}_B(A \otimes B) = \mathrm{Tr}_B \mathrm{Tr}_A(A \otimes B) = \mathrm{Tr}(A)\mathrm{Tr}(B)$$

formally $\mathrm{Tr}_A(\cdot) \equiv (\mathrm{Tr} \otimes \mathbb{I})(\cdot)$ ...

in general

$$\mathrm{Tr}_k \left\{ \bigotimes_{j=1}^{N} A_j \right\} \equiv \mathrm{Tr}(A_k) \cdot A_1 \otimes \ldots \otimes A_{k-1} \otimes A_{k+1} \otimes \ldots \otimes A_N$$

even more generall ($1 \leqslant M \leqslant N$)

$$\mathrm{Tr}_{k_1,\ldots,k_M} \left\{ \bigotimes_{j=1}^{N} A_j \right\} \equiv \prod_{j \in \{k_1,\ldots,k_M\}} \mathrm{Tr}(A_j) \cdot \left\{ \bigotimes_{j \in [N] \setminus \{k_1,\ldots,k_M\}} A_j \right\}$$

partial operations – “trick trace” continued

$$|ab\rangle = |a\rangle \otimes |b\rangle$$

$$\mathrm{Tr}_A\Big\{|ab\rangle\langle cd|\Big\} = \mathrm{Tr}\Big\{|a\rangle\langle c|\Big\} \otimes |b\rangle\langle d| = \langle c|a\rangle \otimes |b\rangle\langle d| = \langle c|a\rangle|b\rangle\langle d|$$

$$\mathrm{Tr}_B\Big\{|ab\rangle\langle cd|\Big\} = |a\rangle\langle c| \otimes \mathrm{Tr}\Big\{|b\rangle\langle d|\Big\} = |a\rangle\langle c| \otimes \langle d|b\rangle = |a\rangle\langle c|\langle d|b\rangle$$

$$\begin{aligned}\mathrm{Tr}\Big\{|ab\rangle\langle cd|\Big\} &= \mathrm{Tr}_A\mathrm{Tr}_B\Big\{|a\rangle\langle c| \otimes |b\rangle\langle d|\Big\} = \mathrm{Tr}\Big\{|a\rangle\langle c|\Big\} \otimes \mathrm{Tr}\Big\{|b\rangle\langle d|\Big\} \\ &= \langle c|a\rangle \otimes \langle d|b\rangle = \langle cd|ab\rangle\end{aligned}$$

partial operations

general partial operations acting on subsystems indexed by $\{k_1, ..., k_M\}$

$$\hat{\mathrm{O}}_{k_1,\ldots,k_M}\left\{\bigotimes_{j=1}^{N} A_j\right\} \equiv \prod_{j\in\{k_1,\ldots,k_M\}} \hat{\mathrm{O}}(A_j)\cdot\left\{\bigotimes_{j\in[N]\setminus\{k_1,\ldots,k_M\}} A_j\right\}$$

- partial trace
- partial transpose
- partial measurement
- partial isometry/unitary operations
- partial perumtation (`SWAP`)
- ...

$\otimes$ product induces a natural block structure
$\Longrightarrow$ a partial operation acts on the distinguished physical subsystems

von Neumann entropy and state purity

entropy = measure of information

- statistical entropy[9] $\mathcal{S}_t = k_B \ln \Omega = -k_B \sum_j p_j \ln p_j$
- information entropy[10] $\mathcal{S}_i = -\sum_j p_j \ln p_j$
- quantum entropy[11] $\mathcal{S}_v = \mathcal{S} = -\mathrm{Tr}\{\rho \ln \rho\}$

linear entropy $\mathcal{S}_L \equiv 1 - \mathrm{Tr}\{\rho^2\} = 1 - \mathcal{P}(\rho)$

$$\begin{cases} \mathcal{P}(|\psi\rangle\langle\psi|) & = 1 & \Longrightarrow & \text{pure state} \\ \mathcal{P}(\rho) & \in \left(\frac{1}{d}, 1\right) & \Longrightarrow & \text{mixed state} \\ \mathcal{P}(\rho_*) & = 1/d & \Longrightarrow & \text{maximally mixed state} \end{cases}$$

$$\begin{cases} \mathcal{S}(|\psi\rangle\langle\psi|) & = 0 & \Longrightarrow & \text{complete information about the state} \\ \mathcal{S}(\rho) & \in (0, \ln d) & \Longrightarrow & \text{statistical mixture} \\ \mathcal{S}(\rho_*) & = \ln d & \Longrightarrow & \text{complete ignorance} \end{cases}$$

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[9] L. Boltzmann, W. Gibbs (1875)
[10] C. Shannon (1948)
[11] J. von Neumann (1927)

quantum state purification

let $\mathcal{H}_A$ be an arbitrary finite-dimensional Hilbert space

$$\forall_{\rho \in \mathcal{H}_A} \; \exists_{\mathcal{H}_B} \; \exists_{|\psi\rangle_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B} \; : \; \rho = \mathrm{Tr}_B |\psi\rangle\langle\psi|_{AB}$$

constructive proof

$$\rho \equiv \sum_{j=1}^{\dim \mathcal{H}_A} p_j |j\rangle\langle j| \qquad : \qquad \{|j\rangle\} = \text{some ONB of } \mathcal{H}_A$$

let $\mathcal{H}_B \equiv \mathcal{H}_A$ and let $|j'\rangle \in \mathcal{H}_B$ be some ONB, then

$$|\psi\rangle \equiv \sum_j \sqrt{p_j} |j\rangle \otimes |j'\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$$

quantum state purification

verification

$$\begin{aligned}
\mathrm{Tr}_B\left\{|\psi\rangle\langle\psi|\right\} &= \mathrm{Tr}_B\left\{\left(\sum_j \sqrt{p_j}|j\rangle \otimes |j'\rangle\right)\left(\sum_k \sqrt{p_k}\langle k| \otimes \langle k'|\right)\right\} \\
&= \mathrm{Tr}_B\left\{\sum_{j,k} \sqrt{p_j p_k}|j\rangle\langle k| \otimes |j'\rangle\langle k'|\right\} \\
&= \sum_{j,k} \sqrt{p_j p_k}|j\rangle\langle k|\delta_{j,k} \\
&= \sum_j p_j|j\rangle\langle j| = \rho
\end{aligned}$$

$$\mathrm{Tr}(M) = \sum_j M_{j,j} = \sum_{j,k} M_{j,k}\delta_{j,k}$$

state purification is a non-unique process!

SVD = singular value decomposition / SD

generalization of the eigenvalue decomposition for a matrix $M \in \mathbb{C}^{m\times n}$

$$M = U\Sigma V^\dagger = \sum_{j=1}^{r(M)} \sigma_j u_j \bar{v}_j \quad : \quad r(M) \leqslant \min\{m,n\}$$

$$\texttt{svd}(M) = \sqrt{\texttt{eig}(MM^\dagger)}$$

Schmidt decomposition

every <u>bipartite</u> pure state $|\psi\rangle_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$, with respect to a chosen basis, can be expressed in Schmidt form as a <u>single</u> sum

$$|\psi\rangle_{AB} = \sum_j \sum_k \alpha_{j,k} |j\rangle_A \otimes |k\rangle_B \qquad \longrightarrow \qquad |\psi\rangle_{AB} = \sum_l \sigma_l |l\rangle \otimes |l\rangle$$

Schmidt decomposition – justification

let $\{|j\rangle\}_{j=1}^{d_A}$, $\{|k\rangle\}_{k=1}^{d_B}$ be ONBs, and let $\alpha = [\alpha_{j,k}] \in \mathbb{C}^{d_A \times d_B}$ be the coefficient matrix of $|\psi\rangle_{AB}$

$$\texttt{svd}(\alpha) = U\Sigma V^\dagger \quad : \quad \begin{cases} U \in \mathbb{U}(d_A) \\ \Sigma = \mathrm{diag}(\sigma_1, \sigma_2, ..., \sigma_r) \in \mathrm{diag}(d_A, d_B) \\ V \in \mathbb{U}(d_B) \end{cases}$$

$$\Longrightarrow \qquad |\psi\rangle_{AB} = \sum_{j=1}^{d_A} \sum_{k=1}^{d_B} \sum_{l=1}^{r} U_{j,l}\sigma_l (V_{l,k})^\dagger |j\rangle \otimes |k\rangle = ...$$

unitary matrices U and V define new ONBs in each subsystem

$$|j'\rangle \equiv \sum_{j=1}^{d_A} U_{j,j'}|j\rangle \qquad \text{and} \qquad |k'\rangle \equiv \sum_{k=1}^{d_B} (V_{k',k})^\dagger |k\rangle$$

primed indices are continuously renamed as l

$$... = \sum_{l=1}^{r} \sigma_l |l\rangle \otimes |l\rangle$$

Schmidt decomposition – possible generalization

there is no direct generalization of SD for multipartite systems! but...

- three- (or more) particle system: $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C \otimes \ldots = ABC\ldots$

  possible bipartitions:
  $$\begin{aligned} \{AB\}\,|\,\{C\} &\longrightarrow \quad |\psi\rangle = \sum \sigma_l^{AB|C} |l\rangle_{AB} |l\rangle_C \\ \{BC\}\,|\,\{A\} &\longrightarrow \quad \ldots \\ \{CA\}\,|\,\{B\} &\longrightarrow \quad \ldots \end{aligned}$$

  Schmidt coefficients $\sigma_l \longrightarrow$ Schmidt vectors
- another approach[12] (LBPS = Local Basis Product States)

  $$\begin{aligned} (\mathbb{C}^2)^{\otimes 3} \ni |\psi\rangle &= \sum_{i,j,k} \tau_{i,j,k} |ijk\rangle \\ &= \lambda_0|000\rangle + \lambda_1 e^{i\varphi}|100\rangle + \lambda_2|101\rangle + \lambda_3|110\rangle + \lambda_4|111\rangle \end{aligned}$$

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[12] explaining why this canonical form is valid requires introducing q. entanglement and local unitary operations, which can shorten the representation of a state without changing the nature of the encoded correlations ⟶ we will discuss this problem when describing entanglement classes for 3-qubit systems: $\{|\mathtt{GHZ}\rangle, |\mathtt{W}\rangle\}$

plan

- Pauli matrices (quantum gates)
- quantum measurement (PVM, POVM)
- quantum entanglement

2×2 Pauli matrices = generators of spin rotations

$$\sigma_0 = \mathtt{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\sigma_x = \mathtt{X} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \sigma_y = \mathtt{Y} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad \sigma_z = \mathtt{Z} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

- Hermitian
- traceless
- basis of (Hermitian) operator space with inner product $\langle A|B\rangle \equiv \mathrm{Tr}\{A^\dagger B\}$
- $\mathtt{I}^2 = \mathtt{X}^2 = \mathtt{Y}^2 = \mathtt{Z}^2 = -i\mathtt{XYZ} = \mathbb{I}_2$
- $\mathtt{ZX} = -\mathtt{XZ} = i\mathtt{Y}$

- generalization of 2×2 matrices $\longrightarrow$ Weyl-Heisenberg group

single system quantum gates

quantum gate ≡ unitary operation

$$\text{identity } \mathtt{I} \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \sum_j |j\rangle\langle j| \quad : \quad \begin{cases} \mathtt{I}|0\rangle = |0\rangle \\ \mathtt{I}|1\rangle = |1\rangle \end{cases}$$

$$\text{bit-flip } \mathtt{X} \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad : \quad \begin{cases} \mathtt{X}|0\rangle = |1\rangle \\ \mathtt{X}|1\rangle = |0\rangle \end{cases}$$

$$\text{bit/phase flip } \mathtt{Y} \equiv \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad : \quad \begin{cases} \mathtt{Y}|0\rangle = i|1\rangle \\ \mathtt{Y}|1\rangle = -i|0\rangle \end{cases}$$

$$\text{phase-flip } \mathtt{Z} \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad : \quad \begin{cases} \mathtt{Z}|0\rangle = |0\rangle \\ \mathtt{Z}|1\rangle = -|1\rangle \end{cases}$$

single system quantum gates

$$\text{Hadamard gate } \mathtt{H} \equiv \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad : \quad \begin{cases} \mathtt{H}|0\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle =: |\mathtt{+}\rangle \\ \mathtt{H}|1\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle =: |\mathtt{-}\rangle \end{cases}$$

$\Longrightarrow$ quantum superposition!

$\Longrightarrow$ transforms computational basis to $\{|\mathtt{+}\rangle, |\mathtt{-}\rangle\}$

$$|\mathtt{-}\rangle\langle\mathtt{-}| + |\mathtt{+}\rangle\langle\mathtt{+}| = \mathbb{I}_2 \quad \text{(ONB)}$$

phase gates:

$$\text{gate } \mathtt{S} \equiv \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \qquad \text{gate } \mathtt{T} \equiv \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix} = \mathtt{R}_z(\pi/4)$$

$$\mathtt{U} = \mathtt{U}(\varphi, \alpha, \beta, \gamma) = e^{i\varphi} \begin{bmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{bmatrix} \begin{bmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{bmatrix} \begin{bmatrix} e^{i\gamma} & 0 \\ 0 & e^{-i\gamma} \end{bmatrix}$$

Pauli matrices 2×2 – spectral decomposition

$$\mathtt{Z} = |0\rangle\langle 0| - |1\rangle\langle 1|$$

$$\mathtt{Z}|0\rangle = +|0\rangle$$
$$\mathtt{Z}|1\rangle = -|1\rangle$$

$$\mathtt{X} = |0\rangle\langle 1| + |1\rangle\langle 0| = |\mathtt{+}\rangle\langle\mathtt{+}| - |\mathtt{-}\rangle\langle\mathtt{-}|$$

$$\mathtt{X}|\mathtt{+}\rangle = +|\mathtt{+}\rangle$$
$$\mathtt{X}|\mathtt{-}\rangle = -|\mathtt{-}\rangle$$

quantum measurement[13]

measurement =

- a process of interaction between a measuring device and the system under observation to obtain information ($\mathbb{R}$) about the system
- quantum: a non-unitary process $\Longrightarrow$ irreversible! (destructive)

typical scenario:

1. state preparation
2. unitary evolution
3. measurement ⟶ transition to the classical domain (interpretation of measurement outcomes)

---

[13] Copenhagen interpretation: N. Bohr vs. D. Bohm, H. Everett III, ...

observable – textbook model

observable = self-adjoint operator (Hermitian matrix)

$$A^\dagger = A : \mathbb{C}^{d\times d} \to \mathbb{C}^{d\times d} \qquad : \qquad 2 \leqslant d < \infty$$

quantum-mechanically measurable physical quantity

{Hamiltonian, position, momentum, orbital angular momentum, spin, charge, ...}

is described by an observable, which in its eigenbasis[14] has diagonal form

$$A = \sum_{j=0}^{d-1} \lambda_j |j\rangle\langle j| \qquad : \qquad 0 \leqslant A \leqslant \mathbb{I}_d \text{ (PSD)}$$

- eigenvalues $\lambda_j \in \mathbb{R}$ are the possible measurement outcomes
- eigenvectors $\{|j\rangle\}_{j=0}^{d-1}$ form an ONB in $\mathbb{C}^d$ (complete set of vectors)

---

[14] spectral theorem

von Neumann measurement postulate

$\{|j\rangle\}$ = ONB = complete set of eigenvectors of some observable $A \in \mathbb{C}^{d\times d}$

$$|\psi\rangle = \sum_j c_j|j\rangle \in \mathbb{C}^d \quad : \quad c_j \in \mathbb{C} \quad \wedge \quad \sum_j |c_j|^2 = 1$$

$\{|k\rangle\}$ = another ONB corresponding to observable B, i.e., $B|k\rangle = \lambda_k|k\rangle$

> as a result of measuring the state $|\psi\rangle$ in the basis $\{|k\rangle\}$, one obtains one of the eigenvalues λ_k of B with probability
>
> $$p_k \equiv |\langle k|\psi\rangle|^2 = \Big|\sum_j c_j\langle k|j\rangle\Big|^2 \text{ and the state collapses } |\psi\rangle \overset{\star}{=} |k\rangle$$

1. measurement performed in the same basis $\Longrightarrow p_k = |c_k|^2$
2. non-selective measurement $\Longrightarrow$

$$|\psi\rangle \longrightarrow \rho \overset{\star}{=} \sum_k p_k|k\rangle\langle k| \quad : \quad |k\rangle\langle k| = \text{projectors defining the measurement}$$

quantum measurement – abstract mathematical definition

quantum measurement:

$$\mathcal{M} = \Big\{M_0, M_1, ..., M_{r-1}\Big\}_{r\in\mathbb{N}_1}$$

$$\forall_j : M_j \in \mathbb{C}^{d\times d} \quad \longleftrightarrow \quad \text{measurement outcome labeled } \textcircled{j}$$

completeness: $\sum_{j=0}^{r-1} M_j^\dagger M_j = \mathbb{I}_d$ (all possible outcomes are accounted for)

in the laboratory, a measurement is identified by a detector

$\Longrightarrow$ assigning natural numbers to measurement outcomes is “just” mathematics

$\Longrightarrow$ any naming convention is allowed to identify measurement outcomes

$$\mathcal{M} = \{M_\uparrow, M_\downarrow\}$$
$$\mathcal{M} = \{M_\otimes, M_\oplus, M_\ominus, M_\odot, M_\oslash\}$$

projective measurement: PVM = projection-valued measure

$$M_j : \begin{cases} M_j = M_j^\dagger \\ M_j = M_j^2 \text{ (idempotent)} \implies M_j \geqslant 0 \text{ (PSD)} \\ M_j M_k = \delta_{jk} M_k \text{ (orthogonality)} \\ \sum_j M_j = \mathbb{I}_d \text{ (resolution of identity)} \end{cases}$$

$$\implies \text{completeness relation: } \sum_j M_j^\dagger M_j = \sum_j M_j^2 = \sum_j M_j = \mathbb{I}_d$$

measurement of a system → probability of obtaining outcome labeled ⓙ

$$\text{Born rule:} \quad p_j = \begin{cases} \langle\psi|M_j|\psi\rangle \\ \\ \mathrm{Tr}\{M_j \rho\} \end{cases} \qquad \text{state update:} \quad \begin{cases} |\psi\rangle \mapsto \frac{M_j|\psi\rangle}{\sqrt{p_j}} \\ \\ \rho \mapsto \frac{M_j \rho M_j}{p_j} \end{cases}$$

generalized measurement: POVM = positive operator-valued measure

$$E_j : \begin{cases} E_j \geqslant 0 \\ \sum_j E_j = \mathbb{I}_d \end{cases}$$

no orthogonality condition, but $\forall_j : E_j = M_j^\dagger M_j$ (Kraus operators)

measurement of a system → probability of obtaining outcome labeled ⓙ

$$\text{Born rule:} \quad p_j = \begin{cases} \langle\psi|M_j^\dagger M_j|\psi\rangle \\ \\ \mathrm{Tr}\{M_j^\dagger M_j \rho\} \end{cases} \qquad \text{state update:} \quad \begin{cases} |\psi\rangle \mapsto \frac{M_j|\psi\rangle}{\sqrt{p_j}} \\ \\ \rho \mapsto \frac{M_j \rho M_j^\dagger}{p_j} \end{cases}$$

Neumark's Dilation Theorem

POVM ⟶ PVM in an extended Hilbert space

quantum measurement – question

$\longrightarrow$ where did the eigenvalues of the measured observable A go?!

PVM

$$A = \sum_j \lambda_j \underbrace{|j\rangle\langle j|}_{\text{projector}} = \sum_j \lambda_j P_j \quad : \quad P_j^2 = P_j P_j = |j\rangle\langle j|j\rangle\langle j| = |j\rangle\langle j| = P_j \mapsto M_j$$

detector $\longleftrightarrow$ projector $\longleftrightarrow$ eigenvalue

$$\text{formally: } \{\lambda_0, \lambda_1, \lambda_2, \ldots\} \hookrightarrow\!\!\!\!\rightarrow \{\otimes, \oplus, \ominus, \ldots\}$$

$\hookrightarrow\!\!\!\!\rightarrow$ = bijection, i.e., a one-to-one correspondence

they are encoded in the projectors of A mapped onto the m. apparatus!

quantum measurement $\longleftrightarrow$ basis

Lecture #`01/15`:

"... a Hilbert space basis = a set of reference vectors ..."

quantum computing:
quantum measurement is (almost) always performed in the computational basis

measurement in the `Z` basis

however, if not $\implies U \in \mathbb{U}(\dots)$ to change the basis...

experiment ⟶ statistics of detector "clicks"

probabilistic nature of quantum measurement outcomes

- single "shot"
- repeated many times

1. initial state preparation
2. unitary evolution of the system
3. quantum measurement

$\Longrightarrow$ statistics $\xrightarrow{\# \to \infty}$ probability density distribution

quantum measurement on state $|\psi\rangle = \tau_0|0\rangle + \tau_1|1\rangle$

consider a quantum measurement in the [eigenvector] Z basis

$$\mathcal{M} = \{M_0, M_1\} = \Big\{|0\rangle\langle 0|, |1\rangle\langle 1|\Big\}$$

we obtain

$$\begin{cases} p_0 & = \langle\psi|M_0|\psi\rangle = \dots = \tau_0^*\tau_0\langle 0|0\rangle\langle 0|0\rangle = \tau_0^*\tau_0 = |\tau_0|^2 \\ |\psi\rangle \mapsto \frac{1}{\sqrt{p_0}}M_0|\psi\rangle & = \frac{1}{\tau_0}|0\rangle\langle 0|(\tau_0|0\rangle + \tau_1|1\rangle) = \frac{1}{\tau_0}\tau_0|0\rangle\langle 0|0\rangle = |0\rangle \end{cases}$$

$$\begin{cases} p_1 & = \langle\psi|M_1|\psi\rangle = \dots = |\tau_1|^2 \\ |\psi\rangle \mapsto \frac{1}{\sqrt{p_1}}M_1|\psi\rangle & = \dots = |1\rangle \end{cases}$$

consistency with von Neumann postulate:

the probability of finding the state in one of the basis states during a measurement in the same basis is given by the squared modulus of the amplitude!

bipartite measurement on state $|\Phi_2^+\rangle$ in two incompatible bases

basis $\mathtt{Z}$: $\mathcal{M}_{\mathtt{Z}} = \{P_{jk} = |jk\rangle\langle jk|\} \;:\; j,k \in \{0,1\}$

basis $\mathtt{X}$: $\mathcal{M}_{\mathtt{X}} = \{P_{jk} = |jk\rangle\langle jk|\} \;:\; j,k \in \{\mathtt{-},\mathtt{+}\}$

in both cases, the measurement is well-defined

$$\sum_{j,k} P_{j,k} = \sum_{j,k} P_{j,k}^2 = \sum_{j,k} |jk\rangle\langle jk| = \ldots = \mathbb{I}_4$$

and has 4 outcomes, symbolically indexed, e.g.

$$\mathcal{M}_{\mathtt{Z}} \longleftrightarrow \{\mathtt{00},\mathtt{01},\mathtt{10},\mathtt{11}\} \quad , \quad \mathcal{M}_{\mathtt{X}} \longleftrightarrow \{\mathtt{--},\mathtt{-+},\mathtt{+-},\mathtt{++}\}$$

1. the actual eigenvalues ∓ 1 of the observables $\mathtt{Z}$ and $\mathtt{X}$ have been mapped onto the above symbol strings
2. we infer from context that this refers to $\mathtt{Z} \otimes \mathtt{Z}$ ...

Pauli matrices <u>do not</u> commute

$\mathtt{XZ} \neq \mathtt{ZX} \quad \Longrightarrow \quad$ the bases are “independent” (incompatible)

bipartite measurement on state $|\Phi_2^+\rangle$ in two incompatible bases

upon measurement in the $\mathtt{Z}$ basis, we obtain

$$p_{\mathtt{00}} = \langle\Phi_2^+|P_{00}|\Phi_2^+\rangle = 1/2 \qquad |\Phi_2^+\rangle \mapsto |00\rangle$$

$$p_{\mathtt{01}} = \langle\Phi_2^+|P_{01}|\Phi_2^+\rangle = 0 \qquad \text{detector 01 does not click}$$

$$p_{\mathtt{10}} = \langle\Phi_2^+|P_{10}|\Phi_2^+\rangle = 0 \qquad \text{detector 10 does not click}$$

$$p_{\mathtt{11}} = \langle\Phi_2^+|P_{11}|\Phi_2^+\rangle = 1/2 \qquad |\Phi_2^+\rangle \mapsto |11\rangle$$

m. in the $\mathtt{X}$ basis is preceded by a unitary rotation of both “parts” of the state $|\Phi_2^+\rangle$

repeat the calculations as for the $\mathtt{Z}$ basis

$$p_{\mathtt{--}} = \langle\Phi_2^+|(\mathtt{H}^\dagger \otimes \mathtt{H}^\dagger)P_{\mathtt{--}}(\mathtt{H} \otimes \mathtt{H})|\Phi_2^+\rangle = 1/2 \qquad |\Phi_2^+\rangle \mapsto |\mathtt{--}\rangle$$

$$p_{\mathtt{-+}} = \langle\Phi_2^+|(\mathtt{H}^\dagger \otimes \mathtt{H}^\dagger)P_{\mathtt{-+}}(\mathtt{H} \otimes \mathtt{H})|\Phi_2^+\rangle = 0 \qquad \text{detector -+ does not click}$$

$$p_{\mathtt{+-}} = \langle\Phi_2^+|(\mathtt{H}^\dagger \otimes \mathtt{H}^\dagger)P_{\mathtt{+-}}(\mathtt{H} \otimes \mathtt{H})|\Phi_2^+\rangle = 0 \qquad \text{detector +- does not click}$$

$$p_{\mathtt{++}} = \langle\Phi_2^+|(\mathtt{H}^\dagger \otimes \mathtt{H}^\dagger)P_{\mathtt{++}}(\mathtt{H} \otimes \mathtt{H})|\Phi_2^+\rangle = 1/2 \qquad |\Phi_2^+\rangle \mapsto |\mathtt{++}\rangle$$

correlations of measurement outcomes $\Longrightarrow$ quantum entanglement

when one of the subsystems obtains a certain measurement outcome, the other subsystem **always** obtains the **same** outcome!

$$p_{jk} = 0 \quad : \quad j \neq k$$

$\Longrightarrow$ complete correlation of the outcomes!

the probability that the outcomes are identical[15] is **1**
however, the marginal probability distribution:

$$\left\{ p_{jk} = |\langle jk|\Phi_2^+\rangle|^2 = \frac{1}{2} \,:\, j = k \right\} \Longrightarrow \textbf{the individual m. results are random!}$$

---

[15] for this particular state! (outcomes can also be perfectly anti-correlated!)

plan

- quantum entanglement – definition and properties
- qubit $\Longrightarrow$ Bloch ball/sphere
- qudit
- multipartite entanglement
- special matrix operations: partial transpose, reshuffling (realignment)
- bipartite entanglement criteria (v. Neumann entropy, SD, $\mathcal{N}$, CCNR, ...)

bipartite quantum entanglement – definition (pure states)

let $|\psi\rangle_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$ be a bipartite q. state

$\exists_{|\phi\rangle_A \in \mathcal{H}_A} \exists_{|\phi\rangle_B \in \mathcal{H}_B} : |\psi\rangle_{AB} = |\phi\rangle_A \otimes |\phi\rangle_B \implies |\psi\rangle_{AB}$ is **separable**

otherwise, the state $|\psi\rangle_{AB}$ is **entangled**

note that dimensions of Hilbert spaces are not specified, it is only assumed that $2 \leqslant \dim \mathcal{H}_X < \infty$

$\bigotimes$-product Hilbert space contains all linear combinations of $\otimes$ product vectors

only a (small) subset of these vectors are **simple** tensors, $a \otimes b$; any vector that cannot be written as a single product is an entangled state

here we assume a bipartite case... nevertheless it applies in general

bipartite quantum entanglement – example

attempt to factorize the state

$$|\Phi_2^+\rangle \equiv \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

$$|\Phi_2^+\rangle = \begin{bmatrix} a \\ b \end{bmatrix} \otimes \begin{bmatrix} x \\ y \end{bmatrix} \quad \Longleftrightarrow \quad \begin{cases} a \cdot x & = 1 \\ a \cdot y & = 0 \\ b \cdot x & = 0 \\ b \cdot y & = 1 \end{cases}$$

gives a contradictory system!

$\Longrightarrow \mathbb{C}^4 \neq \mathbb{C}^2 \otimes \mathbb{C}^2$, i.e. it is <u>not</u> true that:

$$\forall_{|\psi\rangle \in \mathbb{C}^4} \; \exists_{|\phi\rangle_A \in \mathbb{C}^2} \; \exists_{|\phi\rangle_B \in \mathbb{C}^2} \; : \; |\psi\rangle = |\phi\rangle_A \otimes |\phi\rangle_B$$

Bell basis of maximally entangled states

qubits – formally to be defined in ≈ 3 slides...

$$|\Phi_2^{\mp}\rangle \equiv \frac{1}{\sqrt{2}}|00\rangle \mp \frac{1}{\sqrt{2}}|11\rangle$$ perfect correlations

$$|\Psi_2^{\mp}\rangle \equiv \frac{1}{\sqrt{2}}|01\rangle \mp \frac{1}{\sqrt{2}}|10\rangle$$ perfect anticorrelations

entanglement <u>does not depend</u> on the basis choice

correlation type <u>depends</u> on the basis choice

qudits $$|\Phi_d^{+}\rangle = \frac{1}{\sqrt{d}}\sum_{j=0}^{d-1}|jj\rangle$$

Bell basis of maximally entangled states

maximally entangled state

$$|\Phi_2^+\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

does not depend on the basis choice, so it can also be written as

$$|\Phi_2^+\rangle = \frac{1}{\sqrt{2}}|b_0 b_0\rangle + \frac{1}{\sqrt{2}}|b_1 b_1\rangle$$

for an arbitrarily chosen ONB of the form $\mathtt{B} = |b_0\rangle\langle b_0| + |b_1\rangle\langle b_1|$

$\longrightarrow$ the measurement outcome in such a basis is $|b_0 b_0\rangle$ or $|b_1 b_1\rangle$

$\longrightarrow$ $p_{00} = p_{11} = 1/2$

$\longrightarrow$ $p_{01} = p_{10} = 0$

cf. previous lecture

quantum entanglement...

- ... depends on the tensor structure!

$$|\psi\rangle = [1, 1, 1, 0, 0, 0]^{\mathrm{T}} \in \mathbb{C}^6$$

$$|\psi\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^3 \implies |\psi\rangle = |0\rangle \otimes (|0\rangle + |1\rangle + |2\rangle)$$

$$|\psi\rangle \in \mathbb{C}^3 \otimes \mathbb{C}^2 \implies \ldots$$

- ... does not depend on the choice of local basis within a given subspace
- ... is very fragile ⟶ quantum decoherence
- ... in the many-body case it lacks a complete mathematical description (2026)
- ... is a resource used in many quantum protocols
- ... many-body entanglement ⟹ new materials (states of matter)

quantum entanglement...

- ... is a proper subset of quantum correlations

  ⟶ **quantum nonlocality** (second part of the lecture)
- ... is a source of many misunderstandings due to its counterintuitive nature

  systems that are entangled can be described as a whole, whereas a description of their individual components is **not** available!
- ... is often confused with and mistakenly equated to classical correlations

  ⟶ example with two differently colored balls in two boxes
- ... does not violate SR!

  measurement outcomes: perfectly correlated BUT completely random!
- ... will it explain "quantum gravity"?

qudit – nomenclature

$d = 1$ complex scalar field

$d = 2$ qubit = quantum bit

$d = 3$ qutrit

$d = 4$ ququart

⋮

$d = 6$ quhex

⋮

d qudit

⋮

∞ continuous-variable spaces...

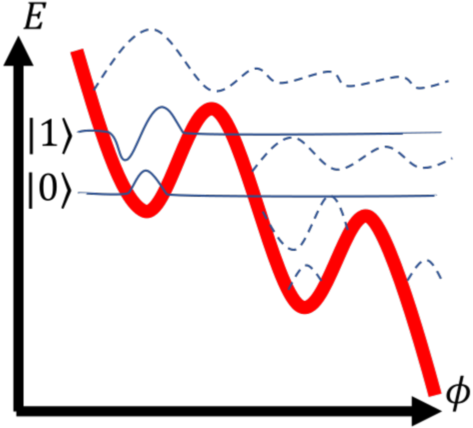

Fig. 1: example physical realization of a qubit

(source: Wikipedia)

qu$\underline{\text{d}}$it – homogeneous systems $(2 \leqslant d < \infty)$

$$\mathbb{C}^d \ni |\psi\rangle = \sum_{j=0}^{d-1} \tau_j |j\rangle \quad : \quad \sum_{j=0}^{d-1} |\tau_j|^2 = 1$$

$$\mathbb{C}^d \otimes \mathbb{C}^d \ni |\psi\rangle = \sum_{j_1=0}^{d-1} \sum_{j_2=0}^{d-1} \tau_{j_1,j_2} |j_1\rangle \otimes |j_2\rangle \quad : \quad \sum_{j_1,j_2=0}^{d-1} |\tau_{j_1,j_2}|^2 = 1$$

$$\underbrace{\mathbb{C}^d \otimes \mathbb{C}^d \otimes \ldots \otimes \mathbb{C}^d}_{=(\mathbb{C}^d)^{\otimes N}} \ni |\psi\rangle = \sum_{j_1=0}^{d-1} \sum_{j_2=0}^{d-1} \cdots \sum_{j_N=0}^{d-1} \tau_{j_1,j_2,\ldots,j_N} |j_1 j_2 \ldots j_N\rangle$$

$$\sum_{j_1,j_2,\ldots,j_N=0}^{d-1} |\tau_{j_1,j_2,\ldots,j_N}|^2 = 1$$

multi-qubit systems $2^{\otimes N}$ (analogy to classical computing)

$$\mathbb{C}^2 \ni |\psi\rangle = \tau_0|0\rangle + \tau_1|1\rangle$$

$$\mathbb{C}^2 \otimes \mathbb{C}^2 \ni |\psi\rangle = \tau_{00}|00\rangle + \tau_{01}|01\rangle + \tau_{10}|10\rangle + \tau_{11}|11\rangle$$

$$\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \ni |\psi\rangle = \tau_{000}|000\rangle + \tau_{001}|001\rangle + \tau_{010}|010\rangle + \tau_{011}|011\rangle + \ldots + \tau_{111}|111\rangle$$

$$(\mathbb{C}^2)^{\otimes N} \ni |\psi\rangle = \tau_{000\ldots0}|000\ldots0\rangle + \ldots + \overbrace{\tau_{\ldots\ldots\ldots}|\underbrace{\ldots\ldots\ldots}_{(k-1)_2}\rangle}^{k\text{-th term}} + \ldots + \tau_{111\ldots1}|111\ldots1\rangle$$

the k-th component in the general expansion of an N-qubit state is indexed by the string representing the binary expansion of consecutive natural numbers from 0 to $2^N - 1$

note that for qutrits the numbering becomes ternary! etc... (base d system)

$$\mathbb{C}^3 \ni |\psi\rangle = \tau_0|0\rangle + \tau_1|1\rangle + \tau_2|2\rangle$$

qudit – heterogeneous systems ($d_1 \neq d_2 \neq \ldots$)

$$\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \ldots \otimes \mathbb{C}^{d_N} \ni |\psi\rangle = \sum_{j_1=0}^{d_1-1} \sum_{j_2=0}^{d_2-1} \ldots \sum_{j_N=0}^{d_N-1} \tau_{j_1,j_2,\ldots,j_N} |j_1 j_2 \ldots j_N\rangle$$

$$\sum_{j_1=0}^{d_1-1} \sum_{j_2=0}^{d_2-1} \ldots \sum_{j_N=0}^{d_N-1} |\tau_{j_1,j_2,\ldots,j_N}|^2 = 1$$

$d = 2 \Longrightarrow$ **qubit:** $|\psi\rangle = \tau_0|0\rangle + \tau_1|1\rangle \quad : \quad |\tau_0|^2 + |\tau_1|^2 = 1$

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle \in \mathbb{C}^2 \qquad : \qquad \theta \in [0, \pi[\qquad \varphi \in [0, 2\pi[$$

global phase is irrelevant: $e^{i\gamma}|\psi\rangle \sim |\psi\rangle$

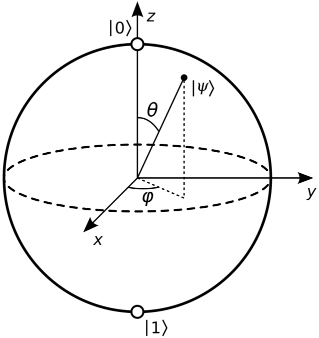

Fig. 2: Bloch sphere[16] – representation of a qubit

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[16] source: Wikipedia

$d = 2 \Longrightarrow$ qubit: $|\psi\rangle = \tau_0|0\rangle + \tau_1|1\rangle \quad : \quad |\tau_0|^2 + |\tau_1|^2 = 1$

- **pure qubits** $|\psi\rangle\langle\psi|$ $\longleftrightarrow$ boundary of the Bloch ball (Bloch sphere)
- **mixed qubits** ρ $\longleftrightarrow$ interior of the Bloch ball
- **maximally mixed qubit** $\rho_* = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\longleftrightarrow$ center of the Bloch ball
- **global phase** γ $\longleftrightarrow$ no physical effect
- **Bloch angles** (φ, θ) $\longleftrightarrow$ a specific qubit state
- **orthogonal qubits** $\longleftrightarrow$ antipodal points on the sphere
- **system of** N **independent qubits** $\longleftrightarrow$ $\times$ product of N Bloch spheres

- $|$**qudit**$\rangle$ $\longleftrightarrow$ an object of size $d^2 - 1$ with a more complicated structure...

$d = 2 \Longrightarrow$ **qubit**: $|\psi\rangle = \tau_0|0\rangle + \tau_1|1\rangle \quad : \quad |\tau_0|^2 + |\tau_1|^2 = 1$

pure state

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle \in \mathbb{C}^2 \qquad : \qquad \theta \in [0, \pi[\qquad \varphi \in [0, 2\pi[$$

corresponding projector

$$\rho = |\psi\rangle\langle\psi| = \Big(\cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle\Big)\Big(\cos\frac{\theta}{2}\langle 0| + e^{-i\varphi}\sin\frac{\theta}{2}\langle 1|\Big) =$$

$$= \begin{bmatrix} \cos^2\frac{\theta}{2} & e^{-i\varphi}\cos\frac{\theta}{2}\sin\frac{\theta}{2} \\ e^{i\varphi}\cos\frac{\theta}{2}\sin\frac{\theta}{2} & \sin^2\frac{\theta}{2} \end{bmatrix}$$

valid density matrix of a <u>pure state</u>

1. Hermitian
2. positive semidefinite (PSD)
3. unit trace (normalized)

$d = 2 \Longrightarrow$ **qubit**: general mixed state

$$\rho = \frac{1}{2}\mathbb{I}_2 + \frac{1}{2}\vec{r}\cdot\vec{\sigma} \qquad \vec{r} = [r_x, r_y, r_z]^{\mathrm{T}} \in \mathbb{R}^3 \longleftarrow \text{Bloch vector}$$

$$= \frac{1}{2}\mathbb{I}_2 + \frac{1}{2}\left(r_x\sigma_x + r_y\sigma_y + r_z\sigma_z\right)$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{2}\left(r_x \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{\sigma_x = \mathtt{X}} + r_y \underbrace{\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}}_{\sigma_y = \mathtt{Y}} + r_z \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{\sigma_z = \mathtt{Z}} \right)$$

$$= \frac{1}{2}\begin{bmatrix} 1 + r_z & r_x - ir_y \\ r_x + ir_y & 1 - r_z \end{bmatrix}$$

valid density matrix of a general qubit state

1. Hermitian
2. positive semidefinite (PSD)
3. unit trace (normalized)

bipartite quantum entanglement

a bipartite pure state $|\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$ is

- **separable/product**, if $|\psi\rangle = |\phi\rangle_1 \otimes |\phi\rangle_2$
- **entangled**, when it is <u>not</u> separable

a bipartite mixed state $\rho \in \mathcal{H}_1 \otimes \mathcal{H}_2$ is

- **separable**, if it is a convex combination of product states

$$\rho = \sum_j q_j \rho_1^{(j)} \otimes \rho_2^{(j)} \quad : \quad \sum_j q_j = 1 \quad \wedge \quad q_j \geqslant 0 \quad \wedge \quad \rho_k \in \mathcal{H}_k$$

  i.e., it contains only classical correlations encoded in the probability distribution $\{q_j\}$

- **entangled**, when it is <u>not</u> separable

bipartite quantum entanglement – example...[17]

note! let

$$\rho \equiv \frac{1}{2}|\Phi_2^+\rangle\langle\Phi_2^+| + \frac{1}{2}|\Phi_2^-\rangle\langle\Phi_2^-| = \dots$$

- the state ρ is a convex combination of maximally entangled states!
- the state ρ is separable

by definition

$\implies$ each matrix $\rho_k^{(j)}$ must belong to a <u>single</u> Hilbert space!

$\implies$ the state ρ turns out to be a combination of product states

$$\dots = \frac{1}{2}|00\rangle\langle00| + \frac{1}{2}|11\rangle\langle11|$$

---

[17] ... of destructive interference that kills entanglement

multipartite quantum entanglement

pure N-partite state

$$|\psi\rangle \in \bigotimes_{k=1}^{N} \mathcal{H}_k \quad \text{is} \quad \begin{cases} \textbf{fully separable/product}, & \text{if } |\psi\rangle = \bigotimes |\phi\rangle_k \\ \textbf{entangled}, & \text{when it is } \underline{\text{not}} \text{ fully separable} \end{cases}$$

mixed N-partite state

$$\rho \in \bigotimes_{k=1}^{N} \mathcal{H}_k \quad \text{is}$$

- **fully separable**, if it is a convex combination of product states

$$\rho = \sum_j q_j \bigotimes_{k=1}^{N} \rho_k^{(j)} \quad : \quad \sum_j q_j = 1 \quad \wedge \quad q_j \geqslant 0 \quad \wedge \quad \rho_k \in \mathcal{H}_k$$

- **entangled**, when it is <u>not</u> fully separable

multipartite quantum entanglement – entanglement hierarchy

- N**-separability** = fully separable
- k**-separability** = (informally) the state can be written as a convex combination of k factors, each belonging to some k-partition (division into k disjoint subsets) of the N Hilbert spaces (each factor in the combination may have a different k-partition!)
- 1**-separability** = the set of all states

- not 2-separable = **genuine multipartite entanglement**[18]
- general lack of separability $\Longrightarrow$ **entanglement**

k-separability $\Longrightarrow$ $(k-1)$-separability

fully separable $\equiv$ N-s. $\subset$ $(N-1)$-s. $\subset$ ... $\subset$ 2-s. $\subset$ 1-s. $\equiv$ set of all states

---

[18] GME = genuine multipartite entanglement

multipartite quantum entanglement – entanglement hierarchy

$\exists$ many other definitions of quantum entanglement that take into account special configurations of the spaces, properties (symmetries), and classes of states

it is suspected that some of these definitions are, from a mathematical point of view, not even properly formulated...

special classes of highly entangled multipartite states:

$\longrightarrow$ $|\mathrm{GHZ}\rangle$ and $|\mathrm{W}\rangle$ states
$\longrightarrow$ AME states[19]
$\longrightarrow$ graph states $\subsetneq$ stabilizer states

---

[19] AME = absolutely maximally entangled – generalization of Bell states

bipartite entanglement criteria – von Neumann entropy

$|\psi\rangle = |00\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$ product pure state
the reduced density matrix of this state is also a pure state (projector)

$$\forall_k : \mathrm{Tr}_k\Big\{|\psi\rangle\langle\psi|\Big\} = |0\rangle\langle 0|$$

$\Longrightarrow$ complete information about the physical system

but

$$\forall_k : \mathrm{Tr}_k\Big\{|\Phi_2^+\rangle\langle\Phi_2^+|\Big\} = \frac{1}{2}\Big(|0\rangle\langle 0| + |1\rangle\langle 1|\Big) = \frac{1}{2}\mathbb{I}_2 = \rho_*$$

i.e., a maximally mixed state

$\Longrightarrow$ complete lack of information about the physical system!

bipartite entanglement criteria – von Neumann entropy

$$S(\rho) \equiv -\mathrm{Tr}\{\rho \log \rho\}$$

$|\psi\rangle_{AB}$ = any 2-qu$\underline{\text{d}}$it state

von Neumann entropy of the reduced density matrix

$$S(|\psi\rangle_{AB}) \equiv S(\rho_A) = S(\rho_B) = -\sum_j \lambda_j \log \lambda_j$$

$$= \begin{cases} 0 & : \ \textbf{separable state} \\ \log d & : \ \textbf{maximally entangled state} \end{cases}$$

λ_j are the eigenvalues of $\rho_k = \mathrm{Tr}_k\{|\psi\rangle_{AB}\langle\psi|\}$

note on the abuse of the symbol S in the definition above!

bipartite entanglement criteria – SD

$$|\psi\rangle_{AB} = \sum_j \sum_k \alpha_{j,k} |j\rangle_A \otimes |k\rangle_B \in \mathcal{H}_A \otimes \mathcal{H}_B \qquad \overset{\text{SD}}{\Longrightarrow}$$

$$|\psi\rangle_{AB} = \sum_l \sigma_l |l\rangle \otimes |l\rangle$$

- $\exists!_{\sigma_l \neq 0} \Longrightarrow$ state is <u>separable</u> ...
- ... otherwise the state is <u>entangled</u>
- $\forall_{j \neq k} : \sigma_j = \sigma_k \neq 0 \Longrightarrow$ state is <u>maximally entangled</u>

special matrix operations – partial transposition

let $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ (∃ natural generalization to N-partite systems)

$$\left.\begin{aligned} (A \otimes B)^{\mathrm{T}_A} &\equiv A^{\mathrm{T}} \otimes B \\[2em] (A \otimes B)^{\mathrm{T}_B} &\equiv A \otimes B^{\mathrm{T}} \end{aligned}\right\} \quad \Longrightarrow \quad \cdot^{\mathrm{T}} = \cdot^{\mathrm{T}_A \mathrm{T}_B} = \cdot^{\mathrm{T}_B \mathrm{T}_A}$$

example: $2 \otimes 3$

$$\mathrm{T}_A : \begin{array}{|ccc|ccc|} \hline & & & a & b & c \\ & & & d & e & f \\ & & & g & h & i \\ \hline a & d & g & & & \\ b & e & h & & & \\ c & f & i & & & \\ \hline \end{array} \qquad \mathrm{T}_B : \begin{array}{|ccc|ccc|} \hline & a & b & & a & b \\ a & & c & a & & c \\ b & c & & b & c & \\ \hline & a & b & & a & b \\ a & & c & a & & c \\ b & c & & b & c & \\ \hline \end{array}$$

$\Gamma \equiv \mathrm{T}_B$

bipartite entanglement criteria – negativity $2 \otimes 2 \vee 2 \otimes 3$ (PPT)

$$\mathcal{N}(\rho) \equiv \frac{||\rho^{\Gamma}||_1 - 1}{2} = \sum_{\lambda_j < 0} |\lambda_j| \qquad : \qquad ||\rho||_1 = \sum \texttt{sv}(\rho)$$

λ_j = **negative** eigenvalues of ρ^{Γ}

the larger the |sum of negative λ's of Γ|, the stronger the entanglement of ρ

$$\rho = |\Phi_2^+\rangle\langle\Phi_2^+| = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \implies \text{eig} = \{0, 0, 0, 1\}$$

$$\rho^{\Gamma} = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \implies \text{eig} = \left\{ -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right\}$$

special matrix operations – reshuffling (realignment)

let $M \in \mathbb{C}^{d\times d}$

$$\text{matrix element } M_{jk} = \langle j|M|k\rangle \quad \Longleftrightarrow \quad M = \sum_{j,k=1}^{d} M_{jk}|j\rangle\langle k|$$

let $d = d_1 d_2$

a bipartite system matrix $M \in \mathbb{C}^{d\times d}$ can be treated as the representation of an operator acting on the composite space $\mathbb{C}^{d_1\times d_1} \otimes \mathbb{C}^{d_2\times d_2}$

$$M_{ij;kl} = \langle ij|M|kl\rangle = \Big(\langle i| \otimes \langle j|\Big) M \Big(|k\rangle \otimes |l\rangle\Big)$$

$$\Longleftrightarrow \quad M = \sum_{i,j,k,l} M_{ij;kl}|i\rangle\langle j| \otimes |k\rangle\langle l| = \sum_{i,j,k,l} M_{ij;kl}\Big(|i\rangle \otimes |k\rangle\Big)\Big(|j\rangle \otimes |l\rangle\Big)$$

multi-index addressing: $M_{ij;kl} = M_{st}$ where

$$\begin{cases} s = j + d_2(i-1) \\ t = l + d_1(k-1) \end{cases}$$

matrix reshuffling (realignment) – example

let $d_1 = d_2 = 2$, i.e., the 2-qubit case

$$M^{\mathrm{R}}_{ij;kl} = M_{ik;jl} \quad \Longleftrightarrow \quad \Big(\langle ij|M|kl\rangle\Big)^{\mathrm{R}} = \langle ik|M|jl\rangle$$

$$\Longleftrightarrow \quad \mathrm{R}\Big(|i\rangle\langle j| \otimes |k\rangle\langle l|\Big) = |i\rangle\langle k| \otimes |j\rangle\langle l|$$

$$M = \left[\begin{array}{cc|cc} M_{1,1} & M_{1,2} & M_{1,3} & M_{1,4} \\ M_{2,1} & M_{2,2} & M_{2,3} & M_{2,4} \\ \hline M_{3,1} & M_{3,2} & M_{3,3} & M_{3,4} \\ M_{4,1} & M_{4,2} & M_{4,3} & M_{4,4} \end{array}\right] \xrightarrow{\mathrm{R}} M^{\mathrm{R}} = \left[\begin{array}{cccc} M_{1,1} & M_{1,2} & M_{2,1} & M_{2,2} \\ \hline M_{1,3} & M_{1,4} & M_{2,3} & M_{2,4} \\ \hline M_{3,1} & M_{3,2} & M_{4,1} & M_{4,2} \\ \hline M_{3,3} & M_{3,4} & M_{4,3} & M_{4,4} \end{array}\right]$$

- blocks $(k) \leftrightarrow$ rows (j)
- only for square matrices $d_1 = d_2$
- involution: $M^{\mathrm{RR}} = M$
- there exists a physical interpretation of this operation

bipartite entanglement criteria – CCNR

= computable cross-norm realignment [O. Rudolph; `arXiv:0202121`]

trace norm of a matrix (Schatten 1-norm, nuclear norm, ...)

$$||M||_1 \equiv \mathrm{Tr}\sqrt{M^\dagger M} = \sum_j \sigma_j$$

then

$$\{\underline{\text{bipartite}}\text{ state } \ \rho \ \text{ is separable}\} \quad \Longrightarrow \quad ||\rho^{\mathrm{R}}||_1 \leqslant 1$$

- CCNR is only a **sufficient** criterion for separability
  $\Longrightarrow$ it does not detect all entangled states!
- detects bipartite entanglement if $||\rho^{\mathrm{R}}||_1 > 1$
- computationally simple entanglement witness (when applicable)
- can be generalized

plan

- $\begin{cases}\text{creation of entanglement – hard} \\ \text{destruction... – easy}\end{cases}$
- quantum circuit
- LOCC paradigm (q. map/instrument formalism required...)
- operational measures of entanglement
- NCT
- quantum teleportation protocol
- abstract measures of entanglement: concurrence, tangle, negativity

two-qubit quantum gates

necessary to generate correlations between two qubit systems

$$\text{gate (control unitary) } \mathtt{CU}_{\mathtt{c;t}} : \begin{cases} \mathtt{c} = \text{control qubit } (\mathtt{control}) \\ \mathtt{t} = \text{target qubit } (\mathtt{target}) \end{cases}$$

$\mathtt{c} = |0\rangle \implies$ no operation on qubit $\mathtt{t}$ (apply identity)

$\mathtt{c} = |1\rangle \implies$ apply gate $\mathtt{U}$ to qubit $\mathtt{t}$

$$\mathtt{CU}_{1;2} \equiv |0\rangle\langle 0| \otimes \mathtt{I} + |1\rangle\langle 1| \otimes \mathtt{U} = \begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & u_{11} & u_{12} \\ \cdot & \cdot & u_{21} & u_{22} \end{bmatrix} \quad \begin{cases} \mathtt{CU}_{1;2}|00\rangle = |00\rangle \\ \mathtt{CU}_{1;2}|01\rangle = |01\rangle \\ \mathtt{CU}_{1;2}|10\rangle = \mathtt{I}|1\rangle \otimes \mathtt{U}|0\rangle \\ \mathtt{CU}_{1;2}|11\rangle = \mathtt{I}|1\rangle \otimes \mathtt{U}|1\rangle \end{cases}$$

$$\mathtt{CU}_{2;1} \equiv \mathtt{I} \otimes |0\rangle\langle 0| + \mathtt{U} \otimes |1\rangle\langle 1| = \begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & u_{11} & \cdot & u_{12} \\ \cdot & \cdot & 1 & \cdot \\ \cdot & u_{21} & \cdot & u_{22} \end{bmatrix} \quad \begin{cases} \mathtt{CU}_{2;1}|00\rangle = |00\rangle \\ \mathtt{CU}_{2;1}|01\rangle = \mathtt{U}|0\rangle \otimes \mathtt{I}|1\rangle \\ \mathtt{CU}_{2;1}|10\rangle = |10\rangle \\ \mathtt{CU}_{2;1}|11\rangle = \mathtt{U}|1\rangle \otimes \mathtt{I}|1\rangle \end{cases}$$

two-qubit quantum gates

necessary to generate correlations between two qubit systems

a special case of the `CU` gate – the `CNOT` gate ($\mathtt{U} = \mathtt{X}$)

$$\mathtt{CNOT}_{\mathtt{1;2}} \equiv |0\rangle\langle 0| \otimes \mathtt{I} + |1\rangle\langle 1| \otimes \mathtt{X} = \begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \end{bmatrix} \qquad \begin{cases} \mathtt{CNOT}_{\mathtt{1;2}}|00\rangle = |00\rangle \\ \mathtt{CNOT}_{\mathtt{1;2}}|01\rangle = |01\rangle \\ \mathtt{CNOT}_{\mathtt{1;2}}|10\rangle = |11\rangle \\ \mathtt{CNOT}_{\mathtt{1;2}}|11\rangle = |10\rangle \end{cases}$$

$$\mathtt{CNOT}_{\mathtt{2;1}} \equiv \mathtt{I} \otimes |0\rangle\langle 0| + \mathtt{X} \otimes |1\rangle\langle 1| = \begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \end{bmatrix} \qquad \begin{cases} \mathtt{CNOT}_{\mathtt{2;1}}|00\rangle = |00\rangle \\ \mathtt{CNOT}_{\mathtt{2;1}}|01\rangle = |11\rangle \\ \mathtt{CNOT}_{\mathtt{2;1}}|10\rangle = |10\rangle \\ \mathtt{CNOT}_{\mathtt{2;1}}|11\rangle = |01\rangle \end{cases}$$

quantum gates – 2-qubit diagram/circuit

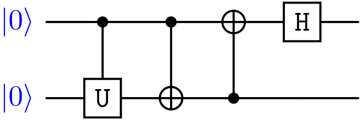


Fig. 3: q. gates: $\mathtt{CU}_{1;2}$, $\mathtt{CNOT}_{1;2}$, $\mathtt{CNOT}_{2;1}$, and $\mathtt{H} \otimes \mathtt{I}$

universal gate set

classical case (remainder)

universal gate `NAND` $\equiv \neg$`AND` (negation of conjunction)
$\Longrightarrow$ possibility to construct any logical operation: {`NOT`, `OR`, `AND`, ...}

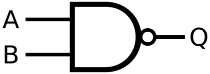


Fig. 4: `NAND` gate = NAND logical operator (source: Wikipedia)

A	B	$Q = \neg(A \wedge B)$
0	0	1
0	1	1
1	0	1
1	1	0

universal gate set ⟶ Gottesman-Knill theorem...

quantum case

$\mathcal{V} = \{\mathcal{V}_1, \mathcal{V}_2, ...\}$ = set of gates from which, with a given precision, one can construct an arbitrary multiqubit unitary operation

$$U \overset{\varepsilon}{\approx} \prod_{j_k} \mathcal{V}_{j_k}$$

$\Longrightarrow$ the set $\mathcal{V}$ guarantees:

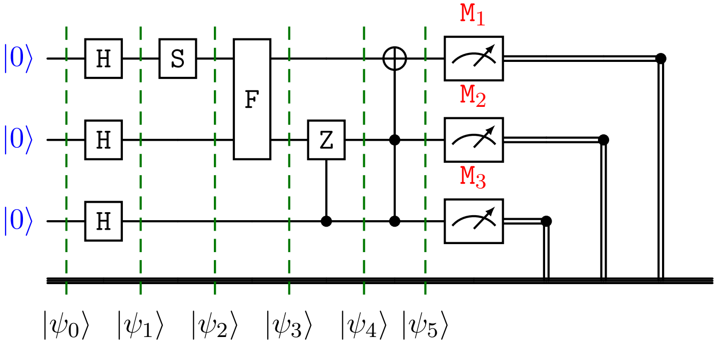
- arbitrary qubit rotation $U \in \mathbb{U}(2)$
- at least one 2-qubit gate ⟶ correlations between qubits

examples[20]:
$\{\mathtt{H}, \mathtt{T}, \mathtt{CNOT}\}$, $\{\mathtt{H}, \mathtt{S}, \mathtt{T}, \mathtt{CNOT}\}$, $\{\mathtt{CZ}, \mathtt{R}_x(\pi/2), \mathtt{R}_z(\pi/4)\}$, $\{\mathtt{CNOT}, \mathtt{R}_x(\theta_x), \mathtt{R}_z(\theta_z)\}$, ...

---

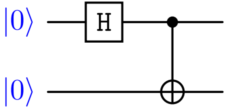
[20] depending on the hardware vendor...

quantum circuit



- quantum circuit: 1. state preparation, 2. unitary evolution, 3. quantum measurement
- single horizontal line = (qubit) quantum bus (serial connection of gates)
- vertical slice ⟶ gates connected in parallel: $\mathtt{U}_{01} = \mathtt{H} \otimes \mathtt{H} \otimes \mathtt{H}$ or $\mathtt{U}_{34} = \mathtt{I} \otimes \mathtt{CZ}_{3;2}$
- unitary evolution: $\mathtt{U} \equiv \mathtt{U}_{45}\mathtt{U}_{34}\mathtt{U}_{23}\mathtt{U}_{12}\mathtt{U}_{01} \in \mathbb{U}(2^3) \Longrightarrow \mathtt{U}|000\rangle = |\psi_5\rangle$
- double line = classical channel carrying the quantum measurement outcome
- quantum measurement (of the state $|\psi_5\rangle$) ⟶ classical computer: control/analysis (post-processing)

quantum circuit generating the maximally entangled state $|\Phi_2^+\rangle$



$$|0\rangle \otimes |0\rangle \xrightarrow{\mathtt{H}\otimes\mathtt{I}} \mathtt{H} \otimes \mathtt{I}|0\rangle \otimes |0\rangle = \mathtt{H}|0\rangle \otimes \mathtt{I}|0\rangle = |+\rangle \otimes |0\rangle$$

$$\xrightarrow{\mathtt{CNOT}_{1;2}} \Big(|0\rangle\langle 0| \otimes \mathbb{I} + |1\rangle\langle 1| \otimes \mathtt{X}\Big)|+\rangle \otimes |0\rangle$$

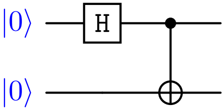
$$= |0\rangle\langle 0|+\rangle \otimes \mathbb{I}|0\rangle + |1\rangle\langle 1|+\rangle \otimes \mathtt{X}|0\rangle$$

$$= \frac{1}{\sqrt{2}}|0\rangle\langle 0|\Big(|0\rangle + |1\rangle\Big) \otimes |0\rangle + \frac{1}{\sqrt{2}}|1\rangle\langle 1|\Big(|0\rangle + |1\rangle\Big) \otimes |1\rangle$$

$$= \frac{1}{\sqrt{2}}|0\rangle\Big(\langle 0|0\rangle + \langle 0|1\rangle\Big) \otimes |0\rangle + \frac{1}{\sqrt{2}}|1\rangle\Big(\langle 1|0\rangle + \langle 1|1\rangle\Big) \otimes |1\rangle$$

$$= \frac{1}{\sqrt{2}}|0\rangle \otimes |0\rangle + \frac{1}{\sqrt{2}}|1\rangle \otimes |1\rangle = \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle\Big) \equiv |\Phi_2^+\rangle$$

quantum circuit generating the maximally entangled state $|\Phi_2^+\rangle$



(algebraic) matrix representation of the circuit

$$\mathtt{CNOT}_{1;2}\big(\mathtt{H}\otimes\mathtt{I}\big)|00\rangle = \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix}1&0&1&0\\0&1&0&1\\0&1&0&-1\\1&0&-1&0\end{bmatrix}}_{\mathtt{CNOT}_{1;2}\cdot\mathtt{H}\otimes\mathtt{I}}\underbrace{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}}_{|00\rangle} = \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\1\end{bmatrix}}_{|\Phi_2^+\rangle}$$

plan

- quantum channel (map)
  - Stinespring representation
  - Kraus representation
  - Choi-Jamiołkowski representation
- (representative) examples
- quantum instrument

evolution of quantum system

isolated system → unitary evolution → Schrödinger equation

non-isolated system[21] → non-unitary evolution → master equation (GKLS)[22]

$$i\hbar\frac{\partial}{\partial t}|\psi_t\rangle = H|\psi_t\rangle \quad \longrightarrow \quad \text{solution: } |\psi_t\rangle = \underbrace{e^{-\frac{i}{\hbar}H(t-t_0)}}_{U(t,t_0)}|\psi_0\rangle$$

$$i\hbar\frac{\partial}{\partial t}\rho_t = [H,\rho_t] \quad \longrightarrow \quad \rho_t = U\rho_0U^\dagger$$

$$|\psi\rangle \longrightarrow U|\psi\rangle$$

$$|\psi_A\rangle \otimes |\psi_B\rangle \otimes \ldots \otimes |\psi_Z\rangle \longrightarrow \overbrace{U_A \otimes U_B \otimes \ldots \otimes U_Z}^{U}|\psi_A\rangle \otimes |\psi_B\rangle \otimes \ldots \otimes |\psi_Z\rangle$$

a unitary operation is reversible: $U^{-1} = U^\dagger$

---

[21] a simplest classification that does not take into account the nature of possible interactions

[22] V. Gorini, A. Kossakowski, G. Lindblad, G. Sudarshan (57th SMP in Toruń!)

quantum channel – motivation for multipartite q. entanglement

- **evolution of entanglement:**
  description of entanglement changes under local/global operations
- **decoherence effects:**
  modeling interaction with environment (entanglement degradation)
- **entanglement detection and manipulation:**
  representation of LOCC protocols, partial operations, q. measurement, etc.
- **error correction and noise modeling:**
  formalization of noise in q. systems

$\Longrightarrow$

unified framework

to describe/control multipartite entanglement in realistic noisy quantum systems

quantum channel (map)

quantum channel

$$\Phi : \mathbb{C}^{d\times d} \ni \rho \mapsto \Phi(\rho) \in \mathbb{C}^{d\times d}$$

= special map
that describes the most general physical evolution of quantum states

1. linearity

$$\forall_{\alpha_j \in \mathbb{C}} : \Phi\Big(\sum_j \alpha_j \rho_j\Big) = \sum_j \alpha_j \Phi(\rho_j)$$

2. $\mathrm{Tr}\{\Phi(\rho)\} = \mathrm{Tr}\{\rho\} = 1$ ⟵ TP = trace preserving

quantum channel

$$S \to S \otimes E$$

environment (heat bath) represented by m-dim. Hilbert space[23]

$$\Phi \quad \longrightarrow \quad \Phi \otimes \mathbb{I}_m : \mathbb{C}^{d\times d} \otimes \mathbb{C}^{m\times m} \to \mathbb{C}^{d\times d} \otimes \mathbb{C}^{m\times m}$$

$\Longrightarrow$ Φ must correctly transform density matrices of a total system:

3. $\forall_{m\geqslant 1} : \Phi \otimes \mathbb{I}_m \geqslant 0$ $\longleftarrow$ CP = completely positive

quantum channel $\equiv$ a map Φ which fulfils 1. – 3.

$$\Phi \in \mathrm{CPTP}$$

---

[23] WLOG: auxiliary system acts on the right (if not – swap it!)

quantum channel – environmental representation (Stinespring, 1955)

every physical process (CPTP) can be modeled as:

$$\begin{aligned}\rho_S \overset{1}{\to} \rho_S \otimes \rho_E &\overset{2}{\to} U_{SE}\Big(\rho_S \otimes \rho_E\Big)U_{SE}^\dagger \\ &\overset{3}{\to} \mathrm{Tr}_E\Big\{U_{SE}\Big(\rho_S \otimes \rho_E\Big)U_{SE}^\dagger\Big\} \overset{4}{=} \Phi(\rho_S)\end{aligned}$$

1. interaction of the system S with the environment represented by the state ρ_E
2. unitary evolution of $S \otimes E$ treated as an $\approx$ isolated system
3. averaging over the environmental degrees of freedom $\longrightarrow$ tracing out E
4. image of the original quantum state under the transformation $\Phi \in$ CPTP

quantum channel – Kraus representation

canonical form of a quantum channel

$$\forall_{\Phi \in \mathrm{CPTP}(\mathbb{C}^{d\times d})} \; \exists_{\{K_j \in \mathbb{C}^{d\times d}\}} \;:\; \Phi(\rho) = \sum_{j=1}^{r} K_j \rho K_j^\dagger$$

$\{K_j\}_{j=1}^{r}$ = set of Kraus operators satisfying the TP condition

$$\sum_{j=1}^{r} K_j^\dagger K_j = \mathbb{I}_d \qquad : \qquad r \leqslant d^2 \text{ (max. environment size: } J_\Phi \ldots)$$

- the form of the channel Φ is environment agnostic
- the form of the Kraus operators **depends** on the environment
- the operators K_j are defined up to an isometry (unitary rotations)

identification of a quantum channel $\iff$ specification of Kraus operators

quantum channel – Kraus representation

the **completeness relation** looks familiar...

$$\sum_j M_j^\dagger M_j = \mathbb{I}_d \quad \Longleftrightarrow \quad \sum_j p_j = 1$$

$$\sum_j K_j^\dagger K_j = \mathbb{I}_d \quad \Longleftrightarrow \quad \text{TP}$$

$\{K_j\}$ are interpreted as measurement operators on E

quantum channel ...

$\longrightarrow$ ... is a black-box (does not track internal stages)

$\longrightarrow$ ... acts as a non-selective measurement (discards m. outcomes)

quantum channel – Kraus representation

any channel can be expressed as a **superoperator**

$$\left.\begin{aligned}\mathtt{vec}(|a\rangle\langle b|) &= |a\rangle \otimes |\bar{b}\rangle \\ \mathtt{vec}(A\rho B^\dagger) &= (A \otimes \bar{B})\mathtt{vec}(\rho)\end{aligned}\right\} \Longrightarrow \mathtt{vec}(\Phi(\rho)) = \Big(\overbrace{\sum_j K_j \otimes \bar{K_j}}^{=\Phi}\Big)\mathtt{vec}(\rho)$$

Kraus representation automatically ensures $\Phi \in$ CPTP

1. linearity: trivial
2. CP: $$(\Phi \otimes \mathbb{I}_m)(\rho) = \sum_j \overbrace{(K_j \otimes \mathbb{I}_m)}^{\equiv \tilde{K}_j}\,\rho\,\overbrace{(K_j^\dagger \otimes \mathbb{I}_m)}^{\equiv \tilde{K}_j^\dagger}$$
3. TP: $$\mathrm{Tr}\{\Phi(\rho)\} = \mathrm{Tr}\Big\{\sum_j K_j\rho K_j^\dagger\Big\} = \sum_j \mathrm{Tr}\{K_j\rho K_j^\dagger\} =$$ $$= \sum_j \mathrm{Tr}\{\rho K_j^\dagger K_j\} = \mathrm{Tr}\Big\{\rho \underbrace{\sum_j K_j^\dagger K_j}_{=\mathbb{I}_d}\Big\} = \mathrm{Tr}\{\rho\}$$

quantum channel – env. representation vs. Kraus form

assume (for simplicity) that env. is in the pure state $|0\rangle_E$

$$\begin{aligned}
&\mathrm{Tr}_E\Big\{U_{SE}\Big(\rho_S \otimes |0\rangle_E\langle 0|\Big)U_{SE}^\dagger\Big\} \\
&= (\mathbb{I}_S \otimes \mathrm{Tr})\Big\{U_{SE}\Big(\rho_S \otimes |0\rangle_E\langle 0|\Big)U_{SE}^\dagger\Big\} \\
&= \sum_e (\mathbb{I}_S \otimes \langle e|)U_{SE}\Big(\rho_S \otimes |0\rangle_E\langle 0|\Big)U_{SE}^\dagger(\mathbb{I}_S \otimes |e\rangle) \\
&= \sum_e (\mathbb{I}_S \otimes \langle e|)U_{SE}(\mathbb{I}_S \otimes |0\rangle_E)\rho_S(\mathbb{I}_S \otimes {}_E\langle 0|)U_{SE}^\dagger(\mathbb{I}_S \otimes |e\rangle) \\
&= \sum_e K_e \rho_S K_e^\dagger
\end{aligned}$$

where $K_e \equiv (\mathbb{I}_S \otimes \langle e|)U_{SE}(\mathbb{I}_S \otimes |0\rangle_E)$

quantum channel – env. representation vs. Kraus form

$$K_e \equiv (\mathbb{I}_S \otimes \langle e|)U_{SE}(\mathbb{I}_S \otimes |0\rangle_E)$$

TP condition:

$$\begin{aligned}
\sum_e K_e^\dagger K_e &= \sum_e \big(\mathbb{I}_S \otimes {}_E\langle 0|\big)U_{SE}^\dagger\big(\mathbb{I}_S \otimes |e\rangle\big)\big(\mathbb{I}_S \otimes \langle e|\big)U_{SE}\big(\mathbb{I}_S \otimes |0\rangle_E\big) \\
&= \sum_e \big(\mathbb{I}_S \otimes {}_E\langle 0|\big)U_{SE}^\dagger\big(\mathbb{I}_S \otimes |e\rangle\langle e|\big)U_{SE}\big(\mathbb{I}_S \otimes |0\rangle_E\big) \\
&= \big(\mathbb{I}_S \otimes {}_E\langle 0|\big)U_{SE}^\dagger \underbrace{\Big(\mathbb{I}_S \otimes \sum_e |e\rangle\langle e|\Big)}_{=\mathbb{I}_{SE}} U_{SE}\big(\mathbb{I}_S \otimes |0\rangle_E\big) \\
&= \big(\mathbb{I}_S \otimes {}_E\langle 0|\big)U_{SE}^\dagger U_{SE}\big(\mathbb{I}_S \otimes |0\rangle_E\big) \\
&= \big(\mathbb{I}_S \otimes {}_E\langle 0|\big)\big(\mathbb{I}_S \otimes |0\rangle_E\big) \\
&= \mathbb{I}_S \otimes {}_E\langle 0|0\rangle_E = \mathbb{I}_S
\end{aligned}$$

quantum channel – Choi-Jamiołkowski representation

each quantum channel

$\Phi : \rho \mapsto \Phi(\rho) \qquad : \qquad \Phi \in \mathrm{CPTP}$

can be associated to a bipartite operator

$$J_\Phi \equiv d(\Phi \otimes \mathbb{I}).|\Phi_d^+\rangle\langle\Phi_d^+| \qquad \text{where} \qquad |\Phi_d^+\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |jj\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$$

1–1 correspondence between q. map Φ and a PSD matrix J_Φ

- CP of $\Phi \iff J_\Phi \geqslant 0$
- TP of $\Phi \iff \mathrm{Tr}_1\{J_\Phi\} = \mathbb{I}$

quantum channel – Choi-Jamiołkowski representation (alternative)

$$J_\Phi = \sum_{j,k=0}^{d-1} \Phi(E_{j,k}) \otimes E_{j,k} \in \mathbb{C}^{d^2 \times d^2}$$

where $\left\{E_{j,k} = |j\rangle\langle k|\right\}_{j,k=0}^{d-1}$ is an ONB (operators) in $\mathbb{C}^{d\times d}$

eigenvectors $|v_j\rangle$ of J_Φ with non-zero eigenvalues λ_j give Kraus operators:

$$K_j = \sqrt{\lambda_j}\,\mathtt{mat}(|v_j\rangle)$$

`mat` is the inverse of vectorization:

$$\mathtt{vec}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a, b, c, d \end{bmatrix}^{\mathrm{T}} \quad : \quad \mathtt{mat}(\mathtt{vec}(M)) = M$$

then

$$\Phi(\rho) = \mathtt{mat}\Big\{\Big(\sum_j K_j \otimes \overline{K}_j\Big)^{\mathrm{R}} \mathtt{vec}(\rho)\Big\}$$

quantum channel – Choi-Jamiołkowski representation

$$\Phi(\rho) = \mathrm{Tr}_1\{(\mathbb{I} \otimes \rho^{\mathrm{T}}) J_\Phi\}$$

$$\Phi(\rho) = \sum_j K_j \rho K_j^\dagger \qquad \Longrightarrow \qquad J_\Phi = \sum_j |K_j\rangle\rangle\langle\langle K_j|$$

$r = \mathrm{rank}(J_\Phi) \leqslant d^2 \quad \Longrightarrow \quad$ max. environment size

quantum channel – generic examples

- partial trace

  trace (fundamental condition is violated)
- unitary evolution
- quantum teleportation protocol
- {LOCC}
- entanglement detection (not always CPTP)
- quantum measurement (in a certain sense...)
- ...

single-qubit examples of quantum channels

recommended: John Watrous' explanation @ `cMl-xIDSmXI`

let $p \in [0,1]$ = probability that a given process error occurs during transmission

bit-flip channel

$$\Phi_{\mathrm{BF}}(\rho) = (1-p)\rho + p\,\mathtt{X}\rho\mathtt{X}$$

Kraus operators: $K_0 = \sqrt{1-p}\,\mathbb{I}_2$ and $K_1 = \sqrt{p}\,\mathtt{X}$

shrinks the yz-plane by a factor $1-2p$ while leaving the x-axis unchanged

phase-flip channel

$$\Phi_{\mathrm{PF}}(\rho) = (1-p)\rho + p\,\mathtt{Z}\rho\mathtt{Z}$$

shrinks the xy-plane by a factor $1-2p$ while leaving the z-axis unchanged

digression: decoherence in quantum systems

decoherence ≡
q. system loses its q. **coherence** due to interaction with the environment (E)

- off-diagonal elements of the density matrix decay ⟶ classical mixtures
- coherent superpositions of states become probabilistic mixtures over time
- basis-dependent: observed in the basis in which the E “probes” the system
- transition from quantum to classical realm (≈ q. measurement)

  leakage of quantum information into the environment...
- modeled by:
  - **phase damping / dephasing** maps ⟶ off-diagonal decay without energy loss
  - **amplitude damping** ⟶ energy relaxation with loss of coherence

⟹ **no** quantum computers as of 2026 :/

phase damping (decoherence) channel

phase damping channel (phase-decohering) in a fixed basis $\{|j\rangle\}$:

$$\Phi_{\mathrm{PD}}(\rho) = \sum_j |j\rangle\langle j|\,\rho\,|j\rangle\langle j|$$

it zeros off-diagonal elements of ρ in this basis

alternative definition using a basis $\mathcal{B} = \{|\psi_j\rangle\}_{j=1,2} \subset \mathbb{C}^2$

$$\Phi_{\mathrm{PD}}(\rho) = (1-p)\rho + p\sum_{j=1}^{2} E_j \rho E_j, \quad E_j = |\psi_j\rangle\langle\psi_j|$$

Kraus operators:

$$K_0 = \sqrt{1-p}\,\mathbb{I}_2, \quad K_1 = \sqrt{p}\,E_1, \quad K_2 = \sqrt{p}\,E_2.$$

again: decoherence depends on the choice of basis!

phase damping channel – action on qubits

consider a general qubit state

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle, \quad \rho = |\psi\rangle\langle\psi| = \begin{bmatrix} \cos^2\frac{\theta}{2} & 2e^{-i\varphi}\sin\theta \\ 2e^{i\varphi}\sin\theta & \sin^2\frac{\theta}{2} \end{bmatrix}$$

and apply Φ_{PD} k times

$$\Phi_{\mathrm{PD}}^k(\rho) = \begin{bmatrix} \cos^2\frac{\theta}{2} & \propto (1-p)^k \\ \propto (1-p)^k & \sin^2\frac{\theta}{2} \end{bmatrix} \xrightarrow{k\to\infty} \begin{bmatrix} \cos^2\frac{\theta}{2} & 0 \\ 0 & \sin^2\frac{\theta}{2} \end{bmatrix}.$$

⟶ off-diagonal elements gradually vanish

⟶ in another basis, decoherence is not observed unless ρ is expressed in that basis

phase damping channel – environment (isometric) representation

use an isometry $U_{SE} : \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^3$

$$U_{SE} : \begin{cases} |0\rangle \to \sqrt{1-p}|0\rangle_S \otimes |0\rangle_E + \sqrt{p}|0\rangle_S \otimes |1\rangle_E \\ |1\rangle \to \sqrt{1-p}|1\rangle_S \otimes |0\rangle_E + \sqrt{p}|1\rangle_S \otimes |2\rangle_E \end{cases}$$

interpretation:
environment $E = \mathbb{C}^3$ is excited to $|1\rangle_E$ or $|2\rangle_E$ with probability p, depending on whether the qubit is in $|0\rangle$ or $|1\rangle$

phase damping channel – Kraus representation

in the computational basis, U_{SE} has the matrix form

$$U_{SE} = \begin{bmatrix} \sqrt{1-p} & 0 \\ \sqrt{p} & 0 \\ 0 & 0 \\ 0 & \sqrt{1-p} \\ 0 & 0 \\ 0 & \sqrt{p} \end{bmatrix}, \quad U_{SE}^{\dagger} U_{SE} = \mathbb{I}_2$$

Kraus operators are obtained via partial trace (check previous slides)

$$K_e = (\mathbb{I}_S \otimes \langle e|) U_{SE}, \quad \{|e\rangle\}_{\mathrm{ONB}} \subset E$$

explicitly

$$K_0 = \sqrt{1-p}\,\mathbb{I}_2, \quad K_1 = \sqrt{p}\,|0\rangle\langle 0|, \quad K_2 = \sqrt{p}\,|1\rangle\langle 1|,$$

which perfectly matches the Kraus representation in the computational basis!

canonical models of physical noise in quantum systems

- **bit-flip** ⟶ simulates bit-flip errors: $|0\rangle \leftrightarrow |1\rangle$
- **phase-flip** ⟶ simulates phase errors: $|1\rangle \to -|1\rangle$
- **bit-phase-flip** ⟶ combined bit/phase flips
- **phase damping/dephasing** ⟶ coherence loss without energy loss (off-diagonal decay)
- **depolarizing** ⟶ uniform noise: qubit $\longrightarrow \rho_\star$
- **amplitude damping** ⟶ energy loss (spontaneous emission...)
- **generalized a. d.** ⟶ finite-T. relaxation with emission/absorption
- **erasure** ⟶ permanent qubit loss
- **Pauli** ⟶ probabilistic combinations of Pauli errors

each channel represents a typical physical error in q. computation/communication

quantum instrument

q. mesurement: produces classical outcomes m and introduces randomness

= a collection of CP maps

$$\{\Phi_m\} \quad : \quad \Phi \equiv \sum_{m \in \mathcal{M}} \Phi_m \in \mathrm{CPTP}$$

indexed by classical outputs (m) of local q. measurements
(performed by parties involved in a protocol)

$\Longrightarrow$ channel Φ describes average evolution of the system
$\Longrightarrow$ instrument additionally tracks individual branches $\{\Phi_m\}$ with classical outputs

quantum instrument as a part of q. protocol

assume a two-party scenario with A and B

1. A measures her portion of a shared state $\rho_{AB} \iff A$ implements an instrument
2. A obtains some classical output $m \iff m$-detector clicks!

$$\rho_{AB} \mapsto \frac{\left(\Phi_m^A \otimes \mathbb{I}_B\right)\rho_{AB}}{p_m = \mathrm{Tr}\ldots}$$

3a. **no postselection** $\Longrightarrow$ A always sends m to B (CC channel)

4a. B applies his operations conditioned on m

$$\rho_{AB} \mapsto \sum_m \left(\Phi_m^A \otimes \Phi_m^B\right)\rho_{AB} \in \text{CPTP as a map} \qquad : \qquad \sum_m \Phi_m^B \in \text{TP}$$

3b. **postselection** $\Longrightarrow$ A keeps only specific outcomes $m \in \tilde{\mathcal{M}}$ discarding others

$$\rho_{AB} \mapsto \frac{\sum_{m\in\tilde{\mathcal{M}}} \left(\Phi_m^A \otimes \Phi_m^B\right)\rho_{AB}}{\sum_{m\in\tilde{\mathcal{M}}} p_m} \notin \text{TP (as a map!)}$$

LOCC = Local Operations & Classical Communication

= the class of local quantum operations on spatially separated systems combined with classical communication (measurement outcomes) between systems/laboratories

strict/axiomatic mathematical definition requires the notion of quantum maps/instruments

- entanglement transformation
  - separable state $\xrightarrow{\text{LOCC}}$ still separable state
  - weakly entangled state $\xrightarrow{\text{LOCC}}$ more entangled state (usually...)
  - maximally entangled state $\xrightarrow{\text{LOCC}}$ any state!
- quantum communication
- distributed quantum computation (quantum networks)
- quantum state discrimination
- multipartite quantum teleportation
- ...

SLOCC = operations succeed only with some probability (S = stochastic)

$\Longrightarrow$ entanglement classes $|\text{GHZ}\rangle$ and $|\text{W}\rangle$...

quantum entanglement measure

$$|\psi_\theta\rangle = \cos\frac{\theta}{2}|00\rangle + \sin\frac{\theta}{2}|11\rangle \qquad \Longrightarrow \qquad \begin{cases} |\psi_0\rangle = |00\rangle \\ \\ |\psi_{\pi/2}\rangle = |\Phi_2^+\rangle \end{cases}$$

- there is no observable for entanglement
  $\Longrightarrow$ entanglement cannot be measured directly

- the difficulty of analyzing entanglement increases with the dimension of the Hilbert space: NP-hard
  $\Longrightarrow$ deciding whether a state is separable or entangled is extremely hard
  $\Longrightarrow$ much easier if the factorization components are known[24]

- there is no universal entanglement measure
  $\Longrightarrow$ a state that is entangled according to one measure may exhibit a different entanglement character according to another measure

---

[24] analogy to factoring an integer into prime factors

quantum entanglement measure – axiomatic approach

$$\mathcal{E} : \mathcal{H} \to [0, \infty[$$

1. $\mathcal{E}$(separable state) $= 0$
2. $\mathcal{E}$ does not increase under LOCC (LOCC-monotone)
   an e. measure should quantify correlations that cannot be achieved classically[25]
3. convexity
4. additivity
5. full additivity
6. ...

∃ operational, geometric and abstract measures of entanglement...

---

[25] LOCC does not generate entanglement hence to create strong entanglement under LOCC, the existence of weakly entangled states is required; in particular, LOCC defines a class of operational entanglement measures

operational entanglement measure (LOCC)

= defined in the context of specific tasks performed under LOCC

scheme (approximations allowed)

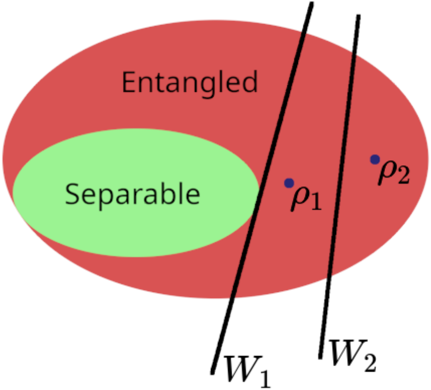
$$\rho_{\texttt{input}}^{\otimes n} \to \Big\{\text{LOCC operations that transform the input state}\Big\} \to \rho_{\texttt{output}}^{\otimes m}$$

1. **entanglement distillation/purification** $\mathcal{E}_D$ quantifies the degree to which weakly entangled states can be converted into highly entangled states
2. **entanglement formation** $\mathcal{E}_F$ quantifies the amount of entanglement (primarily of bipartite systems) required to prepare a given quantum state
3. **entanglement cost** $\mathcal{E}_C$ quantifies the amount of entanglement required to asymptotically prepare many copies of a given state

$$\mathcal{E}_D \leqslant \mathcal{E}_F \leqslant \mathcal{E}_C$$

geometric measure of entanglement[26]

a measure of the distance of a given state from the set of separable states



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[26] detailed considerations of geometric measures go beyond the scope of this lecture

abstract entanglement measures

- entanglement entropy
- concurrence
- tangle
- negativity

concurrence (bipartite qubit systems)

$$|\psi\rangle = \sum_j \sum_k \alpha_{j,k} |j\rangle \otimes |k\rangle \quad : \quad \sum_{j,k} |\alpha_{j,k}|^2 = 1$$

$$\begin{aligned} \mathcal{C}(|\psi\rangle) &\equiv 2|\det \alpha| = |\langle\psi|\mathtt{Y}\otimes\mathtt{Y}|\psi\rangle| \\ &= \sqrt{2\Big(1 - \mathrm{Tr}\{\rho_k^2\}\Big)} = \begin{cases} 0 : \text{non-entangled state} \\ 1 : \text{maximally entangled state} \end{cases} \end{aligned}$$

convex roof extension: $\mathcal{C}(\rho) = \inf_{\{p_j, |\psi_j\rangle\}} \sum_j p_j \mathcal{C}\big(|\psi_j\rangle\big)$

$$\mathcal{C}(\rho) = \max\Big\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\Big\}$$

$$\Big\{\sqrt{\lambda_1} \geqslant \sqrt{\lambda_2} \geqslant \sqrt{\lambda_3} \geqslant \sqrt{\lambda_4}\Big\} = \mathrm{eig}\{\underbrace{\rho(\mathtt{Y}\otimes\mathtt{Y})\bar{\rho}(\mathtt{Y}\otimes\mathtt{Y})}_{\equiv \rho_{\mathtt{Y}}}\} = \mathrm{eig}\Big\{\sqrt{\sqrt{\rho}\rho_{\mathtt{Y}}\sqrt{\rho}}\Big\}$$

∼ fidelity ⟶ II part...

tangle $2 \otimes 2$

$$\tau(|\psi\rangle) \equiv 2\Big(1 - \sum_{j=1}^{d} \sigma_j^2\Big) = \mathcal{C}^2(|\psi\rangle) = 4\sigma_1\sigma_2 \in [0,1]$$

$\{\sigma_j\}$ = Schmidt coefficients of $|\psi\rangle \iff$ eigenvalues of the corresponding RDM

$$\begin{aligned} \tau = 0 \quad &\iff \quad \sigma_k \in \{0,1\}, \text{i.e., } \exists! \text{ non-zero Schmidt coefficient} \\ &\iff \quad \text{state is separable} \end{aligned}$$

$$\begin{aligned} \tau = 1 \quad &\iff \quad \sigma_k = \frac{1}{2} \\ &\iff \quad \text{state is maximally entangled} \end{aligned}$$

tangle $2 \otimes 2 \otimes 2$ (Coffman-Kundu-Wootters monogamy relation)

$$|\psi\rangle_{ABC} = \sum_j \sum_k \sum_l \alpha_{j,k,l} |j\rangle_A \otimes |k\rangle_B \otimes |l\rangle_C \qquad \underbrace{\left\{ \quad A|BC \qquad B|CA \qquad C|AB \quad \right\}}_{\text{Schmidt bipartitions...}}$$

- **1-tangle** = average entanglement w.r.t. three possible partitions

$$\tau_1(|\psi\rangle_{ABC}) \equiv \frac{1}{3}(\tau_{A|BC} + \tau_{B|CA} + \tau_{C|AB})$$

$\tau_{X|YZ} = 4 \det \rho_X = \mathrm{Tr}_{YZ}\{\rho_{XYZ}\} \quad : \quad \rho_{\dots} = |\psi\rangle\langle\psi|_{\dots}$

- **2-tangle** = average entanglement in two-party reductions

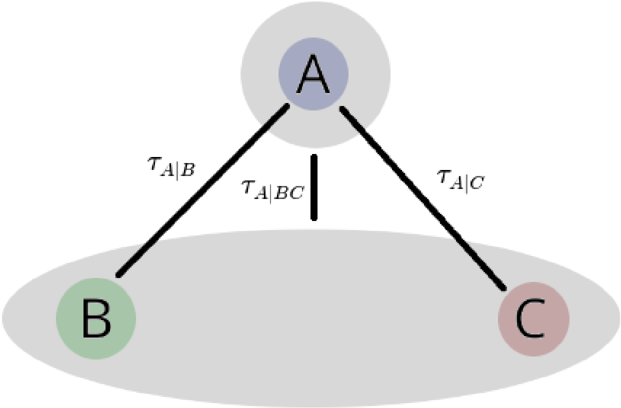
$$\tau_2(|\psi\rangle_{ABC}) \equiv \frac{1}{3}(\tau_{A|B} + \tau_{B|C} + \tau_{C|A})$$

$\tau_{X|Y}$ pertains to $\rho_{XY} = \mathrm{Tr}_Z\{\rho_{XYZ}\} \quad : \quad \rho_{\dots} = |\psi\rangle\langle\psi|_{\dots}$

- **3-tangle** = global entanglement

$$\tau_3(|\psi\rangle_{ABC}) \equiv \tau_{A|BC} - \tau_{A|B} - \tau_{A|C}$$

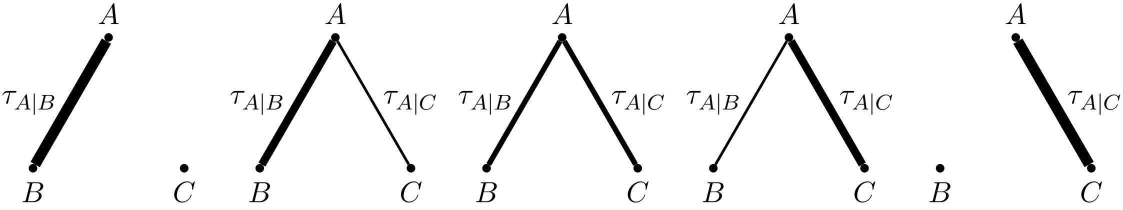
tangle $2 \otimes 2 \otimes 2$



monogamy of entanglement

$$\mathcal{C}^2_{A|B} + \mathcal{C}^2_{A|C} \leqslant \mathcal{C}^2_{A|BC}$$

permutation symmetry: $A \to B \to C \to A$



- qu$\underline{\text{b}}$it systems $\Longrightarrow$ strict relation (max. entanglement is a limited resource)
- qu$\underline{\text{d}}$it systems (CV) $\Longrightarrow$ relaxed condition[27]

---

[27] opposite phenomenon: promiscuity = lubieżność splątania...

provocative question

what is the point of using different measures of entanglement

if **everyone** in the **many-body quantum** community

mostly uses the **von Neumann entropy of entanglement**?

plan

- LOCC in practice
- q. states: $\begin{cases}\text{no cloning}\\ \text{many copies preparation}\\ \text{teleportation}\end{cases}$

no-cloning of arbitrary quantum states (Wojciech H. Żurek)

hypothesis: $\exists_{U \in \mathbb{U}(d)} \forall_{|\psi\rangle \in \mathbb{C}^d} \; : \; U|\psi\rangle \otimes |0\rangle = |\psi\rangle \otimes |\psi\rangle$

let $|\psi_1\rangle$, $|\psi_2\rangle$ be arbitrary qudit states, let us attempt to ...

... clone them $\begin{cases} U|\psi_1\rangle \otimes |0\rangle = |\psi_1\rangle \otimes |\psi_1\rangle \\ U|\psi_2\rangle \otimes |0\rangle = |\psi_2\rangle \otimes |\psi_2\rangle \end{cases}$ and compute the inner product

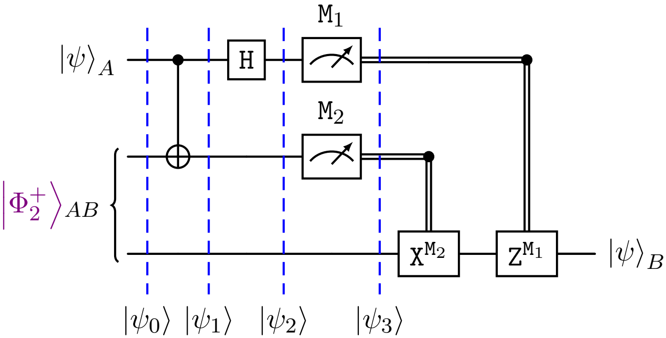
$$\begin{aligned} \big(\langle\psi_1| \otimes \langle\psi_1|\big)\big(|\psi_2\rangle \otimes |\psi_2\rangle\big) &= \langle\psi_1|\psi_2\rangle \otimes \langle\psi_1|\psi_2\rangle = \langle\psi_1|\psi_2\rangle^2 \\ &\overset{U}{=} \big(\langle\psi_1| \otimes \langle 0|\big)U^\dagger U\big(|\psi_2\rangle \otimes |0\rangle\big) \\ &= \big(\langle\psi_1| \otimes \langle 0|\big)\big(|\psi_2\rangle \otimes |0\rangle\big) = \langle\psi_1|\psi_2\rangle \otimes \langle 0|0\rangle = \langle\psi_1|\psi_2\rangle \end{aligned}$$

thus $\langle\psi_1|\psi_2\rangle = 0$ or $\langle\psi_1|\psi_2\rangle = 1 \implies |\psi_1\rangle = |\psi_2\rangle$

> $\implies$ there is no universal quantum operation that can copy arbitrary states
>
> only states from an orthogonal set (basis states) can be perfectly cloned

quantum teleportation protocol of the qubit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

quantum states cannot be cloned, but **any** quantum state (its amplitudes) can be transported over an arbitrary distance[28] using quantum operations and quantum entanglement



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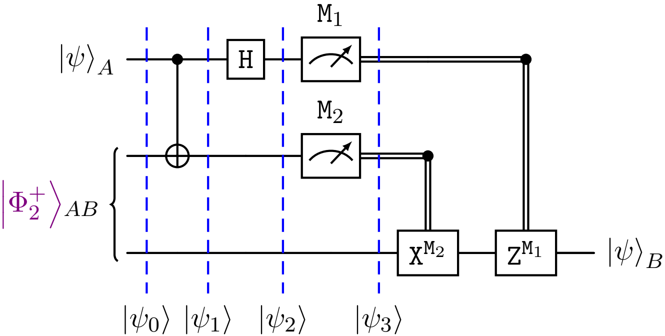
[28] LOCC $\Longrightarrow$ no violation of SR

quantum teleportation protocol of the qubit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$



$$|\psi_0\rangle = |\psi\rangle \otimes |\Phi_2^+\rangle = \frac{1}{\sqrt{2}}\Big(\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle\Big)$$

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}\Big(\alpha|000\rangle + \alpha|011\rangle + \beta|110\rangle + \beta|101\rangle\Big)$$

quantum teleportation protocol of the qubit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

... the Hadamard gate acts on the first qubit:

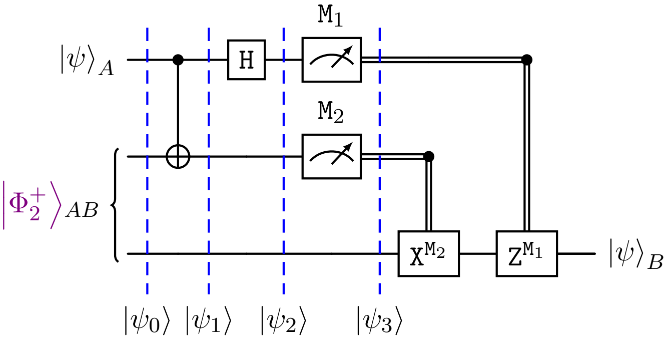
$$
\begin{aligned}
|\psi_2\rangle &= \frac{1}{\sqrt{2}}\Big(\alpha|+\rangle|00\rangle + \alpha|+\rangle|11\rangle + \beta|-\rangle|10\rangle + \beta|-\rangle|01\rangle\Big) = \\
&= \frac{1}{2}\Big(\alpha|000\rangle + \alpha|100\rangle + \alpha|011\rangle + \alpha|111\rangle + \beta|010\rangle - \beta|110\rangle + \beta|001\rangle - \beta|101\rangle\Big) \\
&= \ |00\rangle\big(\alpha|0\rangle + \beta|1\rangle\big)/2 + \\
&\quad\ |01\rangle\big(\alpha|1\rangle + \beta|0\rangle\big)/2 + \\
&\quad\ |10\rangle\big(\alpha|0\rangle - \beta|1\rangle\big)/2 + \\
&\quad\ |11\rangle\big(\alpha|1\rangle - \beta|0\rangle\big)/2
\end{aligned}
$$

one can see the “hidden” state $\alpha|0\rangle + \beta|1\rangle$ appearing in four different configurations

$\longrightarrow$ laboratory A performs a measurement on its qubits and notifies B (LOCC)

$\longrightarrow$ laboratory B applies the appropriate gate in order to correct the amplitudes

quantum teleportation protocol of the qubit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$



measurement in the `Z` basis

– in fact A measures in the Bell basis! hence the operations: $\mathtt{CX}_{1;2}\mathtt{H}_1$

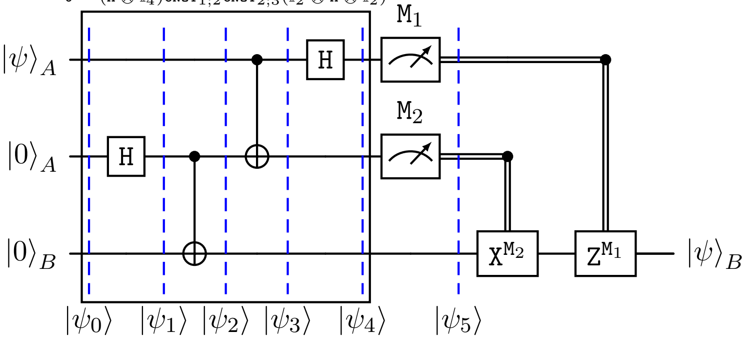
$$\mathcal{M} = \left\{M_{00}, M_{01}, M_{10}, M_{11}\right\} \quad : \quad M_{jk} = |j\rangle\langle j| \otimes |k\rangle\langle k| \otimes \mathbb{I}_2$$

final “state recovery” operation on the side of B

$$|\psi\rangle_B = \mathtt{Z}^{\mathtt{M}_1}\mathtt{X}^{\mathtt{M}_2}|\psi_3\rangle$$

quantum teleportation protocol – alternative circuit

$\mathtt{U} = (\mathtt{H} \otimes \mathbb{I}_4)\mathtt{CNOT}_{1;2}\mathtt{CNOT}_{2;3}(\mathbb{I}_2 \otimes \mathtt{H} \otimes \mathbb{I}_2)$



input state of the system (initial register)

$$|\psi_0\rangle = |\psi\rangle\,|00\rangle = |\psi\rangle_A \otimes |0\rangle_A \otimes |0\rangle_B$$

is the above quantum circuit optimal?
is it really LOCC as it contains two-qubit `CX` gates? :)

quantum teleportation protocol – commentary

- no conflict with the NCT
  the quantum state (arbitrary) initially held by laboratory A collapses as a result of the measurement and ceases to exist, reappearing in laboratory B
- no conflict with SR
  LOCC
- state preparation $\neq$ state copying
  preparing multiple copies of the initial state for repeated experiments implies that the unitary matrix creating the specific state is known, i.e., $|\psi\rangle = U|000...0\rangle$
- quantum states cannot be transmitted classically
  - quantum entanglement is an essential <u>resource</u> for quantum teleportation
  - maximal entanglement $\Longrightarrow$ maximal fidelity of the copied state
- <u>two</u> classical bits and <u>1</u> ebit are required to teleport <u>1</u> qubit
  - multiport-based teleportation $\Longrightarrow$ laboratory B can avoid the state correction...

summary and clarification

$\exists$ three apparently contradictory facts:

$$\begin{cases} \text{preparation of many copies of an arbitrary quantum state: } U|000...0\rangle \mapsto |\psi\rangle, |\psi\rangle, ... \\ \\ \text{no-cloning of arbitrary quantum states: } |\psi\rangle \not\longrightarrow |\psi\rangle, |\psi\rangle, ... \\ \\ \text{quantum teleportation of an arbitrary state: } |\psi\rangle_A \longrightarrow |\psi\rangle_B \end{cases}$$

⟶ they are not contradictory!
⟶ they do not violate STR!
⟶ they form the basis of quantum cryptography

plan

- special classes of (highly entangled) multipartite quantum states
  - $\{|\mathrm{GHZ}\rangle, |\mathrm{W}\rangle\} \in 2 \otimes 2 \otimes 2 = 2^{\otimes 3}$
  - $\mathrm{AME} \in d^{\otimes N}$
  - graph $\subsetneq$ stabilizer states and quantum magic (universal quantum computing)

local unitary equivalence

two pure states in $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \ldots \otimes \mathbb{C}^{d_N}$ are locally unitarily equivalent

$$\Longleftrightarrow$$

$$\exists_{U_1 \in \mathbb{U}(d_1), U_2 \in \mathbb{U}(d_2), \ldots, U_N \in \mathbb{U}(d_1)} : |\psi_1\rangle = U_1 \otimes U_2 \otimes \ldots \otimes U_N |\psi_2\rangle$$

for density matrices: $\rho_1 = \left(U_1 \otimes U_2 \otimes \ldots \otimes U_N\right) \rho_2 \left(U_1 \otimes U_2 \otimes \ldots \otimes U_N\right)^\dagger$

$$|\psi_1\rangle \overset{\texttt{LU}}{=} |\psi_2\rangle$$

note: $\forall_{|\psi_j\rangle} \exists_{U \in \mathbb{U}(d_1 d_2 \cdots d_N)} : |\psi_1\rangle = U |\psi_2\rangle$
i.e., all states are globally unitarily equivalent

states $|\mathrm{GHZ}\rangle$ and $|\mathrm{W}\rangle$ in $2 \otimes 2 \otimes 2$

$$|\mathrm{GHZ}\rangle \equiv \frac{1}{\sqrt{2}}\Big(|000\rangle + |111\rangle\Big)$$

$$|\mathrm{W}\rangle \equiv \frac{1}{\sqrt{3}}\Big(|100\rangle + |010\rangle + |001\rangle\Big)$$

states $|\mathrm{GHZ}\rangle$ and $|\mathrm{W}\rangle$ in $2 \otimes 2 \otimes 2$

- both states are highly, but characteristically, entangled

- the $|\mathrm{GHZ}\rangle$ state generalizes the 2-qubit Bell state $|\Phi_2^+\rangle$

- the two states are **not locally unitarily equivalent**[29]

$$|\mathrm{GHZ}\rangle \overset{\mathrm{LOCC}}{\not\longleftrightarrow} |\mathrm{W}\rangle$$

$\Longrightarrow$ they define two distinct entanglement classes in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$

- any other state can be obtained from either via LOCC

$$\forall_{|\psi\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2} : |\mathrm{GHZ}\rangle \xrightarrow{\mathrm{LOCC}} |\psi\rangle \;\wedge\; |\mathrm{W}\rangle \xrightarrow{\mathrm{LOCC}} |\psi\rangle$$

---

[29] find a global unitary $U \in \mathbb{U}(8) : U|\mathrm{GHZ}\rangle = |\mathrm{W}\rangle$

states $|\mathrm{GHZ}\rangle$ and $|\mathrm{W}\rangle$ in $2 \otimes 2 \otimes 2$

- the partial trace of $|\mathrm{GHZ}\rangle$ over any subsystem yields a product state

  $\Longrightarrow$ genuine multipartite entanglement

- the partial trace of $|\mathrm{W}\rangle$ over any subsystem remains entangled

  $\Longrightarrow$ more robust to loss of a qubit

analogy to Borromean rings

Borromean rings

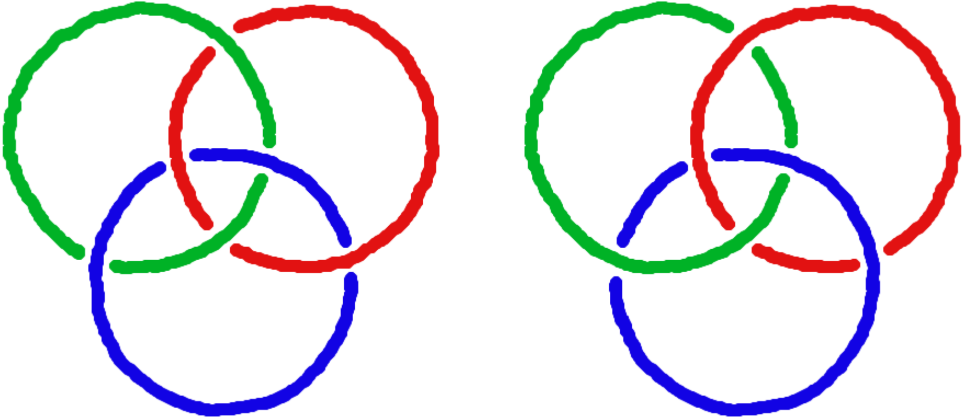


Fig. 5: Borromean rings representing the states $|\mathrm{GHZ}\rangle$ (left) and $|\mathrm{W}\rangle$ (right)

- removing any ring on the left (GHZ) disentangles the remaining rings
- removing any ring on the right (W) still leaves the remaining rings entangled

SD revisited

go back to slide #28

entropy of the RDM of a bipartite maximally entangled state

a state $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$ is maximally entangled when the RDM $\sim \rho_*$

$$|\Phi_d^+\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |jj\rangle \quad \Longrightarrow \quad \rho = |\Phi_d^+\rangle\langle\Phi_d^+| = \frac{1}{d} \left(\sum_{j=0}^{d-1} |jj\rangle \right) \left(\sum_{k=0}^{d-1} \langle kk| \right)$$

$$= \frac{1}{d} \sum_{j,k=0}^{d-1} |j\rangle\langle k| \otimes |j\rangle\langle k|$$

$$\mathrm{Tr}_1\{\rho\} = \frac{1}{d} \sum_{j,k=0}^{d-1} \underbrace{\left(\sum_{l=0}^{d-1} \langle l|j\rangle\langle k|l\rangle \right)}_{=\delta_{lj}\delta_{kl}} \otimes |j\rangle\langle k| = \frac{1}{d} \sum_{l=0}^{d-1} |l\rangle\langle l| = \frac{1}{d}\mathbb{I}_d = \rho_* = \ldots = \mathrm{Tr}_2\{\rho\}$$

thus $$S(\mathrm{RDM}) = -\sum_{1}^{d} \frac{1}{d} \log \frac{1}{d} = -\log \frac{1}{d} = \log d$$

in the case of different local dimensions, $d_1 \neq d_2$, we take $d = \min\{d_1, d_2\}$ $\Longleftarrow$ consequence of SD

absolutely maximally entangled (AME) state

<u>a pure</u> N-partite state

$$|\psi\rangle \in \mathcal{H} = \bigotimes_{k=1}^{N} \mathbb{C}^d = \left(\mathbb{C}^d\right)^{\otimes N} \quad : \quad \dim(\mathcal{H}) = d^N \quad \wedge \quad N \in 2\mathbb{N}$$

is AME(N, d) if for any bipartition of the space $\mathcal{H} = \mathcal{H}_X \otimes \mathcal{H}_{\overline{X}}$, that is <u>balanced</u>, i.e., $\#\{X\} = \#\{\overline{X}\} = N/2$, the RDM is maximally mixed

$$\mathrm{Tr}_{\overline{X}}\{|\psi\rangle\langle\psi|\} = \rho_X = \frac{1}{k}\mathbb{I}_k \quad : \quad k = d^{N/2} = \text{dimension of the partition}$$

⋮

absolutely maximally entangled (AME) state

maximization of the RDM entropy means that for any choice of $N/2$ subsystems

$$
\begin{aligned}
S\left(\frac{1}{k}\mathbb{I}_k\right) &= -\mathrm{Tr}\Big(\rho_X \log \rho_X\Big) = \\
&= -\mathrm{Tr}\begin{bmatrix} \frac{1}{k} & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \frac{1}{k} \end{bmatrix} \log \begin{bmatrix} \frac{1}{k} & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \frac{1}{k} \end{bmatrix} = \\
&= -\mathrm{Tr}\begin{bmatrix} \frac{1}{k}\log\frac{1}{k} & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \frac{1}{k}\log\frac{1}{k} \end{bmatrix} = -\frac{k}{k}\log\frac{1}{k} = -\log\frac{1}{k} = \frac{N}{2}\log d
\end{aligned}
$$

for $N = 2$ we recover the value for the 2-qu$\underline{\text{d}}$it Bell state

AME state with four subsystems – AME$(4, d)$

let

$$|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d \equiv \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C \otimes \mathcal{H}_D$$

$X \subseteq \{A, B, C, D\} \implies$ balanced bipartitions: $AB|CD$, $AC|BD$ and $AD|BC$

$$\text{AME} \qquad \iff \qquad \operatorname{Tr}_{AB}|\psi\rangle\langle\psi| = \operatorname{Tr}_{AC}|\psi\rangle\langle\psi| = \operatorname{Tr}_{AD}|\psi\rangle\langle\psi| = \frac{1}{d^2}\mathbb{I}_{d^2}$$

matrix $U \in \mathbb{U}(d^2)$ is called **2-unitary** iff U^{R} and U^{Γ} are unitary too

theorem: $\exists\, |\psi\rangle \in \text{AME}(4, d) \iff \exists$ 2-unitary matrix $U \in \mathbb{U}(d^2)$

Latin square

classical combinatorics

$$\begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}$$

orthogonal Latin square(s)

two LS are called **orthogonal** (OLS), iff when superimposed, their elements form a unique set of ordered pairs

$$\begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix} \cup \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} a1 & b2 & c3 \\ c2 & a3 & b1 \\ b3 & c1 & a2 \end{bmatrix} \in \mathrm{OLS}(3)$$

it is not always the case:

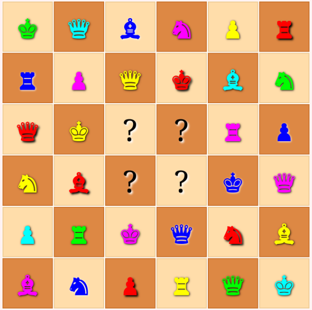
$$\begin{bmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{bmatrix} \cup \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 4 & 3 & 2 & 1 \\ 3 & 4 & 1 & 2 \end{bmatrix} = \begin{bmatrix} a1 & b2 & c3 & d4 \\ d2 & a1 & b4 & c3 \\ c4 & d3 & a2 & b1 \\ b3 & c4 & d1 & a2 \end{bmatrix} \notin \mathrm{OLS}(4)$$

orthogonal Latin square(s) – obstructions

$d = 2$

$$\begin{bmatrix} a\,\square & b\,\square \\ b\,\square & a\,\square \end{bmatrix}$$

$d = 6 \Longrightarrow$ Euler's 36 Officers Problem



orthogonal Latin square(s) – numerical representation

$$\begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix} \cup \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 00 & 11 & 22 \\ 21 & 02 & 10 \\ 12 & 20 & 01 \end{bmatrix}$$

orthogonal Latin square(s) – AME state(s)

quantum combinatorics

$$\begin{bmatrix} 00 & 11 & 22 \\ 21 & 02 & 10 \\ 12 & 20 & 01 \end{bmatrix} \longleftrightarrow \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 00 & 11 & 22 \\ 1 & 21 & 02 & 10 \\ 2 & 12 & 20 & 01 \end{array}$$

two pairs rc and L_{jk} form a unique quartet of numbers, which defines a state[30]

$$\begin{aligned} |\psi\rangle = & \; |0000\rangle + |0111\rangle + |0222\rangle + \\ & \; |1021\rangle + |1102\rangle + |1210\rangle + \\ & \; |2012\rangle + |2120\rangle + |2201\rangle \end{aligned}$$

---

[30] no normalization!

orthogonal Latin square(s) – AME state(s)

proper normalization ($\frac{1}{3}$) together with the qutrit computational basis

$$\left\{ \quad |0\rangle = [1,0,0]^{\mathrm{T}}, \qquad |1\rangle = [0,1,0]^{\mathrm{T}}, \qquad |2\rangle = [0,0,1]^{\mathrm{T}} \quad \right\}$$

makes $|\psi\rangle$ a well-defined state $(. = 0)$

$$|\psi\rangle = \frac{1}{3}[1............1............1.......1...1.........1..........1.........1...1.......]^{\mathrm{T}} \in \mathbb{C}^{81}$$

$$\forall_{\{j,k\} \subsetneq \{A,B,C,D\}} : \mathrm{Tr}_{\{j,k\}}\{|\psi\rangle\langle\psi|\} = \mathbb{I}_9/9 \quad \Longrightarrow \quad |\psi\rangle \in \mathrm{AME}(4,3)$$

theorem: $\forall_{d \neq 2,6}$: $\mathrm{OLS}(d)$ defines a 4-qudit state $\mathrm{AME}(4,d) \in \mathbb{C}^{d^4}$

orthogonal Latin square(s) – permutation matrix

	0	1	2
0	00	11	22
1	21	02	10
2	12	20	01

$\longleftrightarrow$

$$\begin{array}{lll} 00_3 = 0 & 11_3 = 4 & 22_3 = 8 \\ 21_3 = 7 & 02_3 = 2 & 10_3 = 3 \\ 12_3 = 5 & 20_3 = 6 & 01_3 = 1 \end{array}$$

+1 defines a position on unity in a consecutive column of a permutation matrix

$$P_9 \equiv \left[\begin{array}{ccc|ccc|ccc} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}\right]$$

...

orthogonal Latin square(s) – permutation matrix

... this is equivalent to

$$\left.\begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 00 & 11 & 22 \\ 1 & 21 & 02 & 10 \\ 2 & 12 & 20 & 01 \end{array}\right\} \qquad \Longrightarrow \qquad P_9 \equiv \sum_{j=0}^{2} \sum_{k=0}^{2} |L_{jk}\rangle\langle jk| =$$

$$\begin{aligned} &= |00\rangle\langle 00| + |11\rangle\langle 01| + |22\rangle\langle 02| \\ &+ |21\rangle\langle 10| + |02\rangle\langle 11| + |10\rangle\langle 12| \\ &+ |12\rangle\langle 20| + |20\rangle\langle 21| + |01\rangle\langle 22| \end{aligned}$$

where $|L_{jk}\rangle$ denotes a pair of symbols at (j, k)

orthogonal Latin square(s) – permutation matrix

matrix P_9 after reshuffling and partial transposing ...

$$P_9^{\mathrm{R}} = \left[\begin{array}{ccc|ccc|ccc} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \bullet \\ \cdot & \cdot & \cdot & \cdot & \bullet & 1 & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & 1 & \bullet & \cdot & \cdot & \cdot \\ \cdot & \bullet & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \bullet & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \bullet & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}\right] \qquad P_9^{\Gamma} = \left[\begin{array}{ccc|ccc|ccc} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \bullet \\ \cdot & \cdot & \cdot & \cdot & \bullet & \cdot & \cdot & 1 & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \bullet & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \bullet & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \bullet & \cdot \\ \cdot & \cdot & \cdot & \bullet & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}\right]$$

... remains the permutation matrix $\Longrightarrow$ P_9 is 2-unitary!

all of them: $\{P_9, P_9^{\mathrm{R}}, P_9^{\Gamma}\}$ represent $\mathrm{AME}(4,3)$ state

orthogonal Latin square(s) – permutation matrix

for $d \notin \{2, 6\}$
any array $L \in \mathrm{OLS}(d)$ defines a d^2-dimensional 2-unitary permutation matrix

$$P_{d^2} \equiv \sum_{j=0}^{d-1} \sum_{k=0}^{d-1} |L_{jk}\rangle\langle jk| \in \mathbb{U}(d^2)$$

that corresponds to a pure state $|\psi\rangle \in \mathrm{AME}(4, d)$

classification and applications of AME states

- existence and classification of AME(N, d) states
  - $(N = 2, d = 2) \quad \Longrightarrow \quad |\psi\rangle \propto |00\rangle + |11\rangle$
  - $(N = 3, d = 2) \quad \Longrightarrow \quad |\psi\rangle \propto |000\rangle + |111\rangle$
  - $(N = 4, d = 2) \quad \Longrightarrow \quad \emptyset$
  - $(N = 4, d = 6) \quad ...$

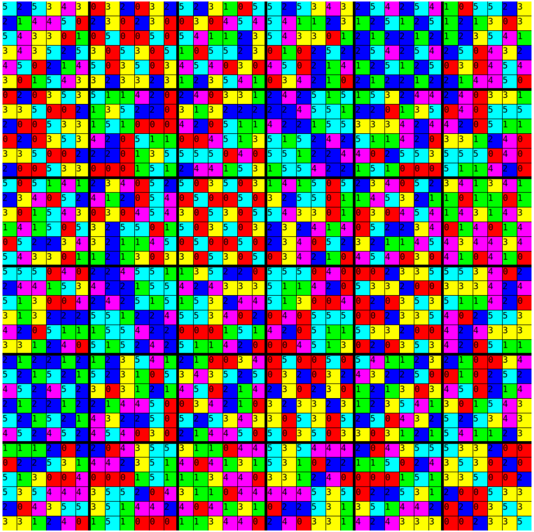
  - Table of AME States by F. Huber and N. Wyderka (soon a new version comes!)

- multipartite quantum state teleportation

- benchmarking of NISQ (Noisy Intermediate-Scale Quantum) devices

- AME vs. graph states $\subsetneq$ stabilizer states

classification of AME states – exotic example



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each $k \in \{0, 1, ..., 5\}$ represents a power of the sixth root of unity $\omega = \exp \frac{2i\pi k}{6}$

plan

- graph states

generalized Pauli matrices

Pauli matrices 2 × 2

$$\mathtt{X} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \mathtt{Y} = i\mathtt{XZ} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad \mathtt{Z} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

let $\{|j\rangle\}_{j=0}^{d-1}$ be an ONB and $\omega = \exp\frac{2\pi i}{d}$

$$\mathtt{Z}|j\rangle \equiv \omega^j|j\rangle \quad \Longleftrightarrow \quad \mathtt{Z} \equiv \sum_{j=0}^{d-1} \omega^j |j\rangle\langle j| = \sum_{j=0}^{d-1} \exp\frac{2\pi i j}{d} |j\rangle\langle j|$$

$$\mathtt{X}|j\rangle \equiv |j \oplus_d 1\rangle \quad \Longleftrightarrow \quad \mathtt{X} \equiv \sum_{j=0}^{d-1} |j \oplus_d 1\rangle\langle j|$$

$$\Longrightarrow \quad \mathtt{ZX} = \omega\mathtt{XZ}$$

generalized Pauli matrices

single qudit Pauli group

$$\mathcal{P}_d \equiv \left\{ \tau^j \mathtt{X}^k \mathtt{Z}^l : j \in \mathbb{Z}_{2d},\ (k,l) \in \mathbb{Z}_d^{\times 2} \right\}_{\mathrm{mod}\ (2d,d)} \quad : \quad \tau = \exp \frac{i\pi(d+1)}{d}$$

N-qudit case

$$\mathcal{P}_d^{\otimes N} \equiv \left\{ \tau^j \bigotimes_{i=1}^{N} \mathtt{X}^{k_i} \mathtt{Z}^{l_i} : j \in \mathbb{Z}_{2d},\ (k_i, l_i) \in \mathbb{Z}_d^{\times 2} \right\}_{\mathrm{mod}\ (2d,d)}$$

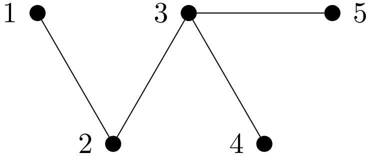
standard Pauli group defined by means of ...

$$\mathcal{P}_2 = \Big\langle \underbrace{\mathtt{X}, \mathtt{Z}, i^k \mathtt{I}}_{\text{... generators}} \Big\rangle = \{\mp\mathtt{I}, \mp\mathtt{X}, \mp\mathtt{Y}, \mp\mathtt{Z}, \mp i\mathtt{I}, \mp i\mathtt{X}, \mp i\mathtt{Y}, \mp i\mathtt{Z}\}$$

graph states

a special multi-qudit class of states

- vertex ⟷ qudit
- edge ⟷ correlations (entanglement)



example of simple (connected) graph with 5 vertices and 4 edges

$$G = (V, E) \quad : \quad \#V = N$$

1. undirected
2. no loops
3. number of edges depends on local dimension d

graph states

qubit

graph states $|G\rangle$ associated with a simple graph $G = (V, E) : \#V = N$

$$|G\rangle \equiv \prod_{\{\mathtt{c},\mathtt{t}\}\in E} \mathtt{CZ}_{\mathtt{c};\mathtt{t}} |+\rangle^{\otimes N} \quad : \quad \begin{cases} |+\rangle = \mathtt{H}|0\rangle \\ \mathtt{CZ}_{\mathtt{c};\mathtt{t}} \equiv |0\rangle\langle 0|_{\mathtt{c}} \otimes \mathtt{I}_{\mathtt{t}} + |1\rangle\langle 1|_{\mathtt{c}} \otimes \mathtt{Z}_{\mathtt{t}} \end{cases}$$

qudit

$$\mathtt{H} \to \mathtt{F} = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} \sum_{k=0}^{d-1} |j\rangle\langle k| \underbrace{\exp\left\{\frac{2\pi i jk}{d}\right\}}_{\omega^{jk}}$$

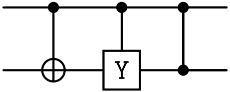
$$\mathtt{CZ}_{\mathtt{c};\mathtt{t}} \equiv \sum_{j=0}^{d-1} \sum_{k=0}^{d-1} \omega^{jk} |j\rangle\langle j|_{\mathtt{c}} \otimes |k\rangle\langle k|_{\mathtt{t}} = \sum_{j=0}^{d-1} |j\rangle\langle j|_{\mathtt{c}} \otimes \mathtt{Z}_{\mathtt{t}}^{j} \quad : \quad \langle jk|\mathtt{CZ}|jk\rangle = \omega^{jk}$$

graph states

graph:

1. undirected $\longleftrightarrow$ edges without arrows $\Longleftarrow$ $\mathtt{CZ}_{\mathtt{c;t}}$ gate is symmetric

   $\cancel{\mathtt{CX}_{\mathtt{c;t}}}$



2. no loops $\Longleftarrow$ $\mathtt{CZ}_{\mathtt{c;t}}$ gate does not act on a single subspace!
3. number of edges is d-dependent

   $\mathtt{Z}^2 = \mathbb{I}_{2\times 2}$ for qubits $\longrightarrow$ $\exists$ edge or no edge

   $\mathtt{Z}^d = \mathbb{I}_{d\times d}$ for qudits $\longrightarrow$ can $\exists$ many $\equiv$ between any 2 vertices

graph states – example

1. separable state $|000\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \longleftrightarrow$ totally disconnected graph
2. superposition

$$\mathtt{F} \otimes \mathtt{F} \otimes \mathtt{F}|000\rangle = |\mathtt{+}\rangle|\mathtt{+}\rangle|\mathtt{+}\rangle$$
$$= \frac{1}{2\sqrt{2}}\Big(|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle\Big)$$

3. edges: correlations induced by gates $\mathtt{CZ}_{\mathtt{c;t}}$

gates $\mathtt{CZ}$ commute (they are diagonal) $\Longrightarrow$ their order is irrelevant

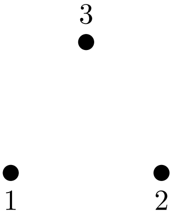
$$\mathtt{CZ}_{1;2}|\mathtt{+++}\rangle \propto \Big(|000 + 001 + 010 + 011 + 100 + 101 - 110 - 111\rangle\Big)$$

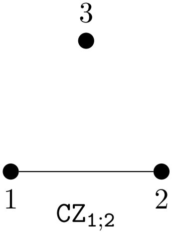
biseparable state

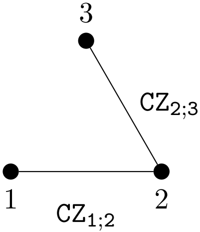
$$\mathtt{CZ}_{2;3}\mathtt{CZ}_{1;2}|\mathtt{+++}\rangle \propto \Big(|000 + 001 + 010 - 011 + 100 + 101 - 110 + 111\rangle\Big)$$
$$\mathtt{CZ}_{1;3}\mathtt{CZ}_{2;3}\mathtt{CZ}_{1;2}|\mathtt{+++}\rangle \propto \Big(|000 + 001 + 010 - 011 + 100 - 101 - 110 - 111\rangle\Big)$$

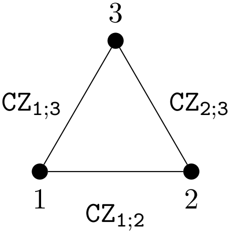
4. we arrive at $|\mathtt{GHZ}\rangle$ – but **not** in the computational basis!

graph states – example



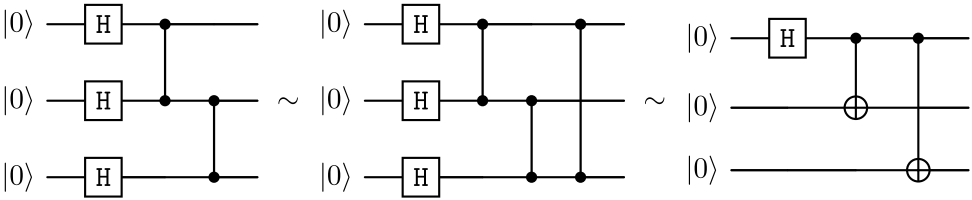




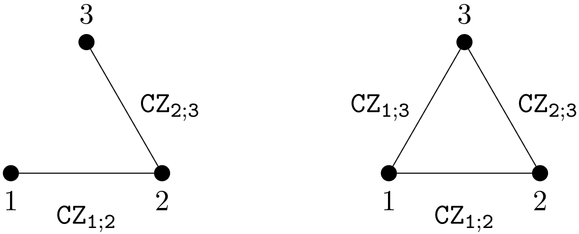


graph states – example

quantum circuit for $|\mathtt{GHZ}\rangle$



all states are LU-equivalent



stabilizer state formalism for graph states

$|G\rangle$ is uniquely characterized as the simultaneous $+1$ eigenstate of a set of commuting operators – stabilizer generators S_j

$$\forall_{j\in V} : S_j \equiv \mathtt{X}_j \bigotimes_{k\in\eta(j)} \mathtt{Z}_k \quad : \quad \eta(j) \equiv \{k : \{j,k\} \in E\} = \text{neigh. of } j$$

hence

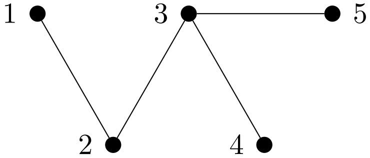
$$\exists!_{|G\rangle} \forall_{j\in V} : S_j|G\rangle = |G\rangle$$

note that this is not the general definition of stabilizer states!

$\Longrightarrow$ efficient classical description of graphs states

- requires only N generators of the stabilizer group
- while N qudits require d^N-dimensional Hilbert space

stabilizer state formalism – example



$$\begin{cases} S_1 = \mathtt{X} \otimes \mathtt{Z} \otimes \mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I} = \mathtt{X}_1\mathtt{Z}_2 \\ S_2 = \mathtt{Z} \otimes \mathtt{X} \otimes \mathtt{Z} \otimes \mathbb{I} \otimes \mathbb{I} = \mathtt{X}_2\mathtt{Z}_1\mathtt{Z}_3 \\ S_3 = \mathbb{I} \otimes \mathtt{Z} \otimes \mathtt{X} \otimes \mathtt{Z} \otimes \mathtt{Z} = \mathtt{X}_3\mathtt{Z}_2\mathtt{Z}_4\mathtt{Z}_5 \\ S_4 = \mathbb{I} \otimes \mathbb{I} \otimes \mathtt{Z} \otimes \mathtt{X} \otimes \mathbb{I} = \mathtt{X}_4\mathtt{Z}_3 \\ S_5 = \mathbb{I} \otimes \mathbb{I} \otimes \mathtt{Z} \otimes \mathbb{I} \otimes \mathtt{X} = \mathtt{X}_5\mathtt{Z}_3 \end{cases}$$

do they commute or not?!

stabilizer state formalism – example

$$|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}|000\rangle + \frac{1}{\sqrt{2}}|111\rangle$$

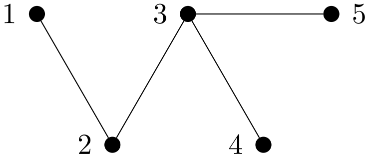
$$\left\{\begin{array}{lcccccc} S_1 & = & \mathtt{X} & \otimes & \mathtt{X} & \otimes & \mathtt{X} \\ S_2 & = & \mathtt{Z} & \otimes & \mathtt{Z} & \otimes & \mathbb{I} \\ S_3 & = & \mathtt{Z} & \otimes & \mathbb{I} & \otimes & \mathtt{Z} \end{array}\right. \qquad \Longrightarrow \qquad \forall_j : S_j|\mathrm{GHZ}\rangle = |\mathrm{GHZ}\rangle$$

graph state – adjacency matrix

$$\Gamma = \Gamma^{\mathrm{T}} \qquad : \qquad \Gamma_{jk} = \begin{cases} l : \text{if vertices } j \text{ and } k \text{ are connected by}\, l \neq 0 \,\text{edges} \\ 0 : \text{otherwise} \end{cases}$$

simple graph has no loops $\implies$ diagonal Γ vanishes[31]

$$|G\rangle = \ldots = \sum_{x\in\{0,1\}^{\otimes N}} \exp\left\{ -i\pi \sum_{j=1}^{N} \sum_{k>j} \Gamma_{jk} x_j x_k \right\} |x\rangle$$



$$\Longleftrightarrow \qquad \Gamma = \begin{bmatrix} \bullet & 1 & \cdot & \cdot & \cdot \\ 1 & \bullet & 1 & \cdot & \cdot \\ \cdot & 1 & \bullet & 1 & 1 \\ \cdot & \cdot & 1 & \bullet & \cdot \\ \cdot & \cdot & 1 & \cdot & \bullet \end{bmatrix}$$

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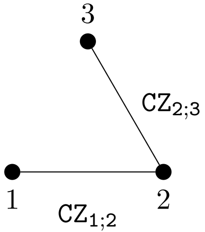
[31] do not be confused with the symbol of partial transpose!

stabilizer state formalism – example



adjacency matrix and stabilizer generators $\mathbb{S} = \langle S_1, S_2, S_3 \rangle$, which ...

$$\Gamma = \begin{bmatrix} \bullet & 1 & \cdot \\ 1 & \bullet & 1 \\ \cdot & 1 & \bullet \end{bmatrix} \quad \Longleftrightarrow \quad \left\{ \begin{array}{lcccccc} S_1 & = & \mathtt{X} & \otimes & \mathtt{Z} & \otimes & \mathbb{I} \\ S_2 & = & \mathtt{Z} & \otimes & \mathtt{X} & \otimes & \mathtt{Z} \\ S_3 & = & \mathbb{I} & \otimes & \mathtt{Z} & \otimes & \mathtt{X} \end{array} \right.$$

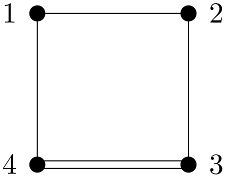
... uniquely identify the state

$$S_j = \mathtt{X}_j \prod_{k=1}^{N=3} \mathtt{Z}_k^{\Gamma_{jk}}$$

graph state: AME$(4,3)$

$$|\mathrm{AME}(4,3)\rangle = \frac{1}{3}\sum_{j=0}^{2}\sum_{k=0}^{2}|j\rangle|k\rangle|j\oplus k\rangle|j\oplus 2k\rangle \in \mathbb{C}^3\otimes\mathbb{C}^3\otimes\mathbb{C}^3\otimes\mathbb{C}^3$$

one of many realizations as a graphs state:



$$\Longleftrightarrow \quad \left\{ \begin{array}{lllllllll} S_1 & = & \mathtt{X} & \otimes & \mathtt{Z} & \otimes & \mathbb{I} & \otimes & \mathtt{Z} \\ S_2 & = & \mathtt{Z} & \otimes & \mathtt{X} & \otimes & \mathtt{Z} & \otimes & \mathbb{I} \\ S_3 & = & \mathbb{I} & \otimes & \mathtt{Z} & \otimes & \mathtt{X} & \otimes & \mathtt{Z}^2 \\ S_4 & = & \mathtt{Z} & \otimes & \mathbb{I} & \otimes & \mathtt{Z}^2 & \otimes & \mathtt{X} \end{array} \right.$$

applications of graph states

- multipartite quantum entanglement $\Longrightarrow$ AME states
- multipartite quantum teleportation protocols
- quantum cryptography (DI QKD[32], ...)
- quantum computing
  - QECC[33]
  - NISQ[34] benchmarking
  - fault-tolerant computation
- quantum networks
- ...

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[32] Device Independent Quantum Key Distribution

[33] Quantum Error Correcting Code(s)

[34] Noisy Intermediate-Scale Quantum

Weyl-Heisenberg and Clifford group

stabilizer Pauli group is a special case of more general Weyl-Heisenberg group

in finite dimensional case both structures are equivalent up to a phase factor

Clifford group $\equiv$ **normalizer** of the Pauli group in the unitary group, i.e.,

$$U\mathcal{P}_d^{\otimes N}U^\dagger = \mathcal{P}_d^{\otimes N}$$

formally

$$\mathcal{C}_d(N) = \mathcal{N}(\mathcal{P}_d^{\otimes N}) \equiv \left\{U \in \mathbb{U}(d^N) : \forall_{P\in\mathcal{P}_d^{\otimes N}} : UPU^\dagger \in \mathcal{P}_d^{\otimes N}\right\}$$

$\iff$ U normalizes the Pauli group up to phase factors

very informally: Cliffords map Paulis to Paulis

(general) stabilizer state definition

a pure N-qudit state $|\psi\rangle \in (\mathbb{C}^d)^{\otimes N}$ is a stabilizer state iff
$\exists$ a maximal abelian subgroup $\mathcal{S} \leqslant \mathcal{P}_d^{\otimes N}$ such that $\tau^j \mathbb{I} \notin \mathcal{S}$ for all $j \in \mathbb{Z}_{2d} \backslash \{0\}$
and $g|\psi\rangle = |\psi\rangle$ for all $g \in \mathcal{S}$

equivalently: $|\psi\rangle$ is the unique common eigenvector with eigenvalue 1 of $\mathcal{S}$

can be generalized to mixed states...

$$|\Phi_2^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \notin \{\texttt{graph states...}\}$$

$$\begin{cases} g_1 = \mathtt{X} \otimes \mathtt{X} = \mathtt{X}_1\mathtt{X}_2 \\ g_2 = \mathtt{Z} \otimes \mathtt{Z} = \mathtt{Z}_1\mathtt{Z}_2 \end{cases}$$

... really?! $\longrightarrow$ $|\Phi_2^+\rangle$ is LC-equivalent to a graph state ...

local Clifford equivalence

$\{$ graph states $\}$ $\subsetneq$ $\{$ stabilizer states $\}$

fact: any stabilizer state is locally Clifford-equivalent to some graph state

$$\forall_{|\psi\rangle \in \{\text{qubit}^{\otimes N} \text{ stabilizer}\}} \exists_{\text{local}\, U_C \in \mathcal{C}_d(N)} : U_C|\psi\rangle = |G\rangle$$

Q: what is the point to say that the Bell state does not belong to graph states?
A: check the form of the stabilizers...

algebraic motivation

let $\{A_j\}_{j=1}^{k}$ = commuting operators $\Longrightarrow$ simultaneously diagonalizable

if $\exists!_{|\psi\rangle} \forall_j : A_j|\psi\rangle = \lambda_j|\psi\rangle$, then the joint eigenspace $\longleftrightarrow$ λ_j is 1-dim.

$\Longrightarrow$ a stabilizer state is the unique 1-eigenstate of its stabilizer generators

$\Longleftrightarrow$ the joint eigenspace is 1-dimensional

$\Longrightarrow$ to stabilize a state uniquely one needs
a maximal commuting set of operators with a 1-dim. joint eigenspace

stabilizer formalism
efficiently encodes commutation relations in the pure linear algebra terms

Gottesman–Knill theorem and ...

a quantum circuit built only from:

- stabilizer states
- Cliffords (mapping stabilizer states to stabilizer states)
- Pauli measurements

is efficiently classically simulable

$\Longrightarrow$ they are not universal!

$\Longrightarrow$ they do not form a dense subgroup of $\mathbb{SU}(d)$

… quantum magic $\Longrightarrow$ universal quantum computing

informally: non-stabilizer states are magic states

example:

$$|T\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{i\pi/4}|1\rangle\right)$$

$\Longrightarrow$ non-stabilizer quantum states enable implementation of a non-Clifford gates

$\Longrightarrow$ quantum magic is a necessary resource for universal quantum computing

Part II

Introduction to Quantum Nonlocality